Frequency-dependent shear viscosity, sound velocity, and sound attenuation near the critical point in liquids. I. Theoretical results

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We calculate the hydrodynamic shear and sound mode near the liquid-gas phase transition and predict its values for different frequencies and temperatures. The dynamical parameters that enter the theoretical expressions are apart from static quantities the background values of the Onsager coefficients for the order parameter and the transverse momentum current. We find within the field-theoretical renormalization group formalism the asymptotic scaling functions or the real and imaginary parts of the shear and sound mode in the ω-ξ^{-1} plane. Our main concern, however, is the calculation of the nonasymptotic expressions describing the crossover from the analytic background to the asymptotic critial behavior. [S1063-651X(97)05412-3]

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I. INTRODUCTION

The dynamical critical behavior of pure fluids near the critical point belongs to the universality class of model H [1]. The dynamics of this model is described by the order parameter, which is chosen as the entropy density, and the transverse momentum current. Thus this model contains the heat conducting mode and the shear mode. However, critical effects can be seen in other modes too, specifically in the sound mode. Measurements of the critical sound attenuation $\alpha(t)$ at the liquid-gas critical point T_c in the limit of zero frequency show a divergence at T_c, which according to theory follows the power law $\alpha(t)\sim\omega^2 t^{-\rho}$ with ω the frequency and $t=(T-T_c)/T_c$ the relative temperature distance at zero frequency. In the asymptotic region the dynamical critical exponents ρ is related to the dynamical critical exponent z, the exponent ν of the correlation length, and the exponent of the specific heat α by $\rho=z\nu+\alpha/2$. The ratio of the sound attenuation amplitudes above and below T_c is given by a universal value [2,3] and is related to the ratio of the amplitudes of the specific heat at zero frequency. However, at finite frequencies the sound attenuation reaches a finite value at T_c leading to a critical frequency dependence at T_c. Thus one has to calculate the attenuation as a function of both variables, temperature and frequency, in the limit of small wave vectors k. Since the experimental temperature region covers the whole region from background to the asymptotic region in the comparison with experiment a nonasymptotic calculation seems to be appropriate. It is the topic of this paper to present the details of such a calculation. It will be seen that the nonasymptotic attenuation at different finite frequencies can be predicted without introducing nonuniversal dynamical parameters apart from those already contained in model H (apart from some subleading terms related to the Onsager coefficient of the bulk viscosity). Those parameters appearing in the attenuation can be taken from the description of the other modes of the liquid within model H. In particular they have been found by comparing the shear viscosity with the theoretical predictions of model H and have already been used to predict the thermal conductivity and the sound attenuation [4]. In this way the sound mode constitutes an important further test of phase transition theories.

A first successful phenomenological approach has been developed in [5]. Asymptotic properties at the critical point in fluids have been studied in [3] within an extension of the dynamical equations of model H [1] by renormalization group theory providing a basis for [5]. A nonasymptotic theory of the sound propagation has been elaborated for the superfluid transition in ^{4}He [6,7], and our approach for the calculation of the sound attenuation near the critical point at T_c in pure fluids is quite in the spirit of this treatment.

In addition to the sound attenuation and sound velocity we calculate the expression of the frequency-dependent viscosity. This has been considered years ago within a decoupled mode theory by Bhattacharjee and Ferrell [8,9]. Here we extend the calculation to the nonasymptotic region.

A derivation of the complete set of equations describing the critical dynamics of a mixture including the sound mode has been given in [10]. Here we use the specification of the set of these equations to the case of pure fluids to calculate within the field theoretic theory of dynamical critical phenomena the sound attenuation in one loop order. Although the attenuation can be calculated by a so-called frequency-dependent specific heat within the simpler model H, we prefer to stay within the complete model and express the attenuation by the appropriate vertex functions. This allows us to identify additional (however, less diverging) terms missing in the simplified version.

The paper is organized as follows. In Sec. II we present the dynamical equations and identify the static vertex functions. Then we relate in Sec. III the hydrodynamic transport coefficients to the unrenormalized dynamic vertex functions and give their one loop expressions. In Sec. IV we perform the necessary renormalizations and show that no new singularities besides those of model H appear. Then we need to relate the transport coefficients to the renormalized vertex functions and to the renormalized dynamical parameters. This is done in Sec. V. The zero frequency results are discussed in Sec. VI and Sec. VII contains the asymptotic scaling functions together with the frequency dependence at T_c. In three Appendixes we present the static and dynamic func-

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tionals used for the loop expansion and give a short description of the derivation of the dynamic equations. A comparison with experiment using the theoretical results will be presented in a second part (a short account on this has been published in [4,11]).

II. EXTENDED DYNAMIC MODEL IN PURE LIQUIDS

In this section we present the dynamic model for the gas-liquid transition suitable for the field-theoretic renormalization group treatment, by which we calculate critical effects in the thermal conductivity, the shear viscosity (frictional coefficient), and the sound propagation. The dynamical universality class is defined by the dynamical equations for the set of local volume densities enclosing the entropy density $s(x)$ and the transverse momentum density $\boldsymbol{j}_t(x)$, with the constraint $\boldsymbol{\nabla}\cdot\boldsymbol{j}_t=0$. Choosing the entropy per mass $\sigma(x)=s(x)/\rho(x)$ as the order parameter the dynamic model that is known as model H was suggested by Halperin, Hohenberg, and Siggia [1,12]. All critical singularities connected with the gas-liquid transition in the pure fluid are related to the critical singularities found within this model. In order to obtain a model that also describes critical sound propagation, the set of densities has to be extended by the mass density $\rho(x)$ and the longitudinal momentum density $\boldsymbol{j}_l(x)$ $(\boldsymbol{\nabla}\times\boldsymbol{j}_l=\boldsymbol{0})$. As a consequence it is necessary to use an extended set of dynamic equations which include mass conservation and the effects of bulk viscosity.

Recently this extended system of dynamic equations and the corresponding static functional has been derived for liquid mixtures [10]. The dynamic model for pure liquids is obtained from these equations by reducing the mixture model and some details of [10] are repeated in the Appendix. The resulting dynamic equations for the order parameter ϕ_0 [the entropy density, defined by Eq. (A14)], the secondary density q_0 [related to the mass density and defined by Eq. (A15)], and the longitudinal and transverse momentum density $\boldsymbol{j}_l,\boldsymbol{j}_t$ are

$$\frac{\partial\phi_0}{\partial t}=\mathring{\Gamma}\nabla^2\frac{\delta H}{\delta\phi_0}+\mathring{L}_\phi\nabla^2\frac{\delta H}{\delta q_0}-\mathring{g}(\boldsymbol{\nabla}\phi_0)\cdot\frac{\delta H}{\delta\boldsymbol{j}}+\Theta_\phi, \tag{2.1}$$

$$\frac{\partial q_0}{\partial t}=\mathring{L}_\phi\nabla^2\frac{\delta H}{\delta\phi_0}+\mathring{\lambda}\nabla^2\frac{\delta H}{\delta q_0}-\mathring{c}\boldsymbol{\nabla}\cdot\frac{\delta H}{\delta\boldsymbol{j}_l}-\mathring{g}\boldsymbol{\nabla}\cdot\left(q_0\frac{\delta H}{\delta\boldsymbol{j}}\right)-\mathring{g}_l\phi_0\boldsymbol{\nabla}\cdot\frac{\delta H}{\delta\boldsymbol{j}_l}+\Theta_q, \tag{2.2}$$

$$\frac{\partial\boldsymbol{j}_l}{\partial t}=\mathring{\lambda}_l\nabla^2\frac{\delta H}{\delta\boldsymbol{j}_l}-\mathring{c}\boldsymbol{\nabla}\cdot\frac{\delta H}{\delta q_0}-\mathring{g}_l\boldsymbol{\nabla}\cdot\left(\phi_0\frac{\delta H}{\delta q_0}\right)+\mathring{g}(1-\mathcal{T})\times\left\{(\boldsymbol{\nabla}\phi_0)\cdot\frac{\delta H}{\delta\phi_0}+q_0\boldsymbol{\nabla}\cdot\frac{\delta H}{\delta q_0}\right\}-\mathring{g}(1-\mathcal{T})\times\left\{\sum_k\left[j_k\boldsymbol{\nabla}\cdot\frac{\delta H}{\delta j_k}-\nabla_k\boldsymbol{j}\frac{\delta H}{\delta j_k}\right]\right\}+\boldsymbol{\Theta}_l, \tag{2.3}$$

$$\frac{\partial\boldsymbol{j}_t}{\partial t}=\mathring{\lambda}_t\nabla^2\frac{\delta H}{\delta\boldsymbol{j}_t}+\mathring{g}\mathcal{T}\left\{(\boldsymbol{\nabla}\phi_0)\cdot\frac{\delta H}{\delta\phi_0}+q_0\boldsymbol{\nabla}\cdot\frac{\delta H}{\delta q_0}\right\}-\mathring{g}\mathcal{T}\left\{\sum_k\left[j_k\boldsymbol{\nabla}\cdot\frac{\delta H}{\delta j_k}-\nabla_k\boldsymbol{j}\frac{\delta H}{\delta j_k}\right]\right\}+\boldsymbol{\Theta}_t. \tag{2.4}$$

$\mathcal{T}$ is the projector to the direction of the transverse momentum density, which corresponds to a projection orthogonal to the wave vector in Fourier space. In the fast fluctuating forces $\Theta_i(x,t)(i=\phi,q,l,t)$ memory effects are irrelevant and their Gaussian spectrum fulfills the Einstein relations,

$$\langle\Theta_i(x,t)\Theta_j(x',t')\rangle=2L_{ij}(x)\,\delta(t-t')\,\delta(x-x'), \tag{2.5}$$

where the matrix $[L_{ij}]$ is given by

$$[L_{ij}]=\begin{pmatrix}-\mathring{\Gamma}\nabla^2 & -\mathring{L}_\phi\nabla^2 & 0 & 0\\ -\mathring{L}_\phi\nabla^2 & -\mathring{\lambda}\nabla^2 & 0 & 0\\ 0 & 0 & -\mathring{\lambda}_l\nabla^2 & 0\\ 0 & 0 & 0 & -\mathring{\lambda}_t\nabla^2\end{pmatrix}. \tag{2.6}$$

The mode couplings $\mathring{c}$, $\mathring{g}$, and $\mathring{g}_l$ are defined as

$$\mathring{c}=RT\rho,\quad \mathring{g}=\frac{RT}{\sqrt{N_A}},\quad \mathring{g}_l=\left(\frac{\partial\rho}{\partial\sigma}\right)_P\frac{RT}{\sqrt{N_A}}, \tag{2.7}$$

with the gas constant R and the Avogadro number N_A. Due to mass conservation the dynamic equation for the mass density is purely reversible (continuity equation). Therefore only three of the five Onsager coefficients $\mathring{\Gamma},\mathring{L}_\phi,\mathring{\lambda},\mathring{\lambda}_l$, and $\mathring{\lambda}_t$ constitute an independent set of coefficients. The coefficients $\mathring{L}_\phi$ and $\mathring{\lambda}$ formally appear because the secondary density q_0 represents a linear combination of the entropy density and the mass density fluctuations. Both of these coefficients are related to $\mathring{\Gamma}$

$$\mathring{L}_\phi=-\left(\frac{\partial\rho}{\partial\sigma}\right)_P\mathring{\Gamma},\quad \mathring{\lambda}=\left(\frac{\partial\rho}{\partial\sigma}\right)_P^2\mathring{\Gamma}. \tag{2.8}$$

In the noncritical background Eqs. (2.1)–(2.4) reduce at vanishing mode couplings to the usual hydrodynamic equations. The independent Onsager coefficients are related to the background values of the thermal conductivity $\kappa_T^{(0)}$, the shear viscosity $\overline{\eta^{(0)}}$, and the bulk viscosity $\zeta^{(0)}$:

$$\mathring{\Gamma}=\frac{R\kappa_T^{(0)}}{\rho^2},\quad \mathring{\lambda}_l=RT\left(\zeta^{(0)}+\frac{4}{3}\overline{\eta^{(0)}}\right),\quad \mathring{\lambda}_t=RT\,\overline{\eta^{(0)}}. \tag{2.9}$$

The choice (A14) and (A15) for order parameter and secondary density guarantees a static functional H with a diagonal Gaussian part and with vanishing third order terms for the order parameter. The corresponding Hamiltonian is

$$H=\int d^dx\left\{\frac{1}{2}\mathring{\tau}\phi_0^2(x)+\frac{1}{2}[\boldsymbol{\nabla}\phi_0(x)]^2+\frac{\mathring{\tilde{u}}}{4!}\phi_0^4(x)\right.$$
$$+\frac{1}{2}a_q q_0^2(x)+\frac{1}{2}\mathring{\gamma}_q q_0(x)\phi_0^2(x)+\frac{1}{2}a_j\boldsymbol{j}_t^2(x)+\frac{1}{2}a_j\boldsymbol{j}_l^2(x)$$
$$\left.-\mathring{h}_q q_0(x)\right\}, \tag{2.10}$$

where a constant external field $\mathring{h}_q$ has been introduced, whose value fixes the average value of the secondary density $q_0(x)$ to zero. Thus in the unordered phase above the critical temperature T_c, where we have $\mathring{\tau}>0$, the expectation value of the order parameter vanishes. The coefficients a_q and a_j are related to background parameters by

$$a_q=\frac{1}{RT\rho}\left(\frac{\partial P}{\partial\rho}\right)_\sigma^{(0)},\qquad a_j=\frac{1}{RT\rho}. \tag{2.11}$$

The superscript (0) in Eq. (2.11) indicates thermodynamic background derivatives that do not contain any critical singularity. The Fourier transforms of the fluctuation density correlation functions taken at vanishing wave number $\boldsymbol{k}$ are related to thermodynamic derivatives. We define

$$\langle AB\rangle_c\equiv\langle AB\rangle_c(k=0)=\int d^dx\langle A(x)B(0)\rangle_c$$
$$=\int d^dx\langle\Delta A(x)\Delta B(0)\rangle_c, \tag{2.12}$$

where A,B hold for $\phi_0,q_0,\boldsymbol{j}_l,\boldsymbol{j}_t$. The subscript c characterizes the cumulant $\langle AB\rangle_c=\langle AB\rangle-\langle A\rangle\langle B\rangle$. The expectation values are taken at vanishing external field variations, $\langle F(A,B,\ldots)\rangle=\int\mathcal{D}\{A,B,\ldots\}F(A,B,\ldots)w_{\rm loc}|_{\delta\mathcal{H}=0}$, with the local probability density $w_{\rm loc}$ defined in Eq. (A1). The static correlations of the order parameter and the secondary densities are related to thermodynamic derivatives via the relations

$$\langle\phi_0\phi_0\rangle_c=\frac{RT}{\rho}\left(\frac{\partial\sigma}{\partial T}\right)_P, \tag{2.13}$$

$$\langle q_0q_0\rangle_c=RT\rho\left(\frac{\partial\rho}{\partial P}\right)_\sigma. \tag{2.14}$$

Equations (2.13) and (2.14) constitute the connection between the static model defined by the functional (2.10) and the thermodynamic derivatives. The momentum density term in (2.10) represents the kinetic energy density, which is isotropic. The correlations of the currents can be written as

$$\langle\boldsymbol{j}_l\otimes\boldsymbol{j}_l\rangle_c=\langle\boldsymbol{j}_t\otimes\boldsymbol{j}_t\rangle_c\equiv\langle jj\rangle_c\mathbf{1} \tag{2.15}$$

in which $\otimes$ denotes the tensor product between two vectors and $\mathbf{1}$ the unit matrix. Because the Hamiltonian does not include interactions of the currents with other densities the correlation (2.15) simply is

$$\langle jj\rangle_c=RT\rho=\frac{1}{a_j}. \tag{2.16}$$

All the nondiagonal correlations vanish and therefore the static two-point vertex functions $\mathring{\Gamma}_{\alpha\beta}$ are simply given by the inverse correlations

$$\mathring{\Gamma}_{\phi\phi}=\frac{1}{\langle\phi_0\phi_0\rangle_c},\quad \mathring{\Gamma}_{qq}=\frac{1}{\langle q_0q_0\rangle_c},\quad \Gamma_{ll}=\Gamma_{tt}=\frac{1}{\langle jj\rangle_c}, \tag{2.17}$$

which may be calculated in a systematic perturbation expansion by accumulating the one-particle irreducible two point graphs [13]. A static functional of the same structure as Eq. (2.10) appears in the critical theory of the superfluid transition in ^{4}He (there the order parameter is a complex quantitiy), therefore we will take over the results known for that case. The static correlation functions are well studied and for calculation details we refer to [14]. The secondary densities $q_0,\boldsymbol{j}_l$, and $\boldsymbol{j}_t$ in (2.10) may be eliminated by integration. Then the static asymptotic behavior in fluids at the gas-liquid transition is completely determined by the ϕ^4 model [1]

$$H=\int d^dx\left\{\frac{1}{2}\mathring{r}\phi_0^2(x)+\frac{1}{2}[\nabla\phi_0(x)]^2+\frac{\mathring{u}}{4!}\phi_0^4(x)\right\}, \tag{2.18}$$

with the parameters

$$\mathring{r}=\mathring{\tau}+\frac{\mathring{\gamma}_q\mathring{h}_q}{a_q},\qquad \mathring{u}=\mathring{\tilde{u}}-\frac{3\mathring{\gamma}_q^2}{a_q}. \tag{2.19}$$

For this reason all vertex functions and correlations in the extended model (2.10) may be written as functions of the ϕ^4-model parameters $\mathring{r},\mathring{u}$ instead of $\mathring{\tau},\mathring{\tilde{u}}$ and the correlations of the secondary density q_0 are related to order parameter correlations calculated with Eq. (2.18). In particular the following relations hold for the expectation value and the two point correlation of the secondary density:

$$\langle q_0\rangle(\mathring{r},\mathring{\gamma}_q,\mathring{u})=\mathring{h}_q-\mathring{\gamma}_q\langle\tfrac{1}{2}\phi_0^2\rangle(\mathring{r},\mathring{u}), \tag{2.20}$$

$$\langle q_0q_0\rangle_c(\mathring{r},\mathring{\gamma}_q,\mathring{u})=a_q+\mathring{\gamma}_q^2\langle\tfrac{1}{2}\phi_0^2\tfrac{1}{2}\phi_0^2\rangle_c(\mathring{r},\mathring{u}). \tag{2.21}$$

From Eq. (2.20) it follows that the constant external field has to be $\mathring{h}_q=\mathring{\gamma}_q\langle\frac{1}{2}\phi_0^2\rangle(\mathring{r},\mathring{u})$ to guarantee a vanishing expectation value of q_0 and relation (2.19) can be written as

$$\mathring{r}=\mathring{\tau}+\frac{\mathring{\gamma}_q^2}{a_q}\langle\tfrac{1}{2}\phi_0^2\rangle(\mathring{r},\mathring{u}), \tag{2.22}$$

which defines the connection between $\mathring{r}$ and $\mathring{\tau}$ in every order of the perturbation expansion.

III. MODEL TRANSPORT COEFFICIENTS

From Eqs. (2.1)–(2.4) we derive a dynamic functional analogous to [15] from which the dynamic two-point vertex functions $\mathring{\Gamma}_{\alpha,\tilde{\beta}}$ are calculated within a Feynman graph expansion. Some definitions and details are presented in the Appendixes. In this section we derive the relations between the dynamic vertex functions of the model and hydrodynamic transport coefficients like thermal conductivity κ_T, shear viscosity $\overline{\eta}$, bulk viscosity ζ, sound velocity c_s, and sound attenuation α. From the hydrodynamic equations for a simple liquid [16] one gets immediately equations linear in the currents

$$\frac{\partial\sigma(x,t)}{\partial t}-\frac{\kappa_T}{\rho T}\nabla^2 T(x,t)=0, \tag{3.1}$$

$$\frac{\partial \boldsymbol{j}_t'(x,t)}{\partial t}-\frac{\overline{\eta}}{\rho}\nabla^2\boldsymbol{j}_t'(x,t)=\mathbf{0}, \tag{3.2}$$

$$\frac{\partial\rho(x,t)}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}_l'(x,t)=0, \tag{3.3}$$

$$\frac{\partial \boldsymbol{j}_l'(x,t)}{\partial t}+\boldsymbol{\nabla}P(x,t)-\frac{1}{\rho}\left(\zeta+\frac{4}{3}\overline{\eta}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{j}_l'(x,t))=\mathbf{0}. \tag{3.4}$$

Introducing the Fourier components ($A=\sigma,\rho,P,T,\boldsymbol{j}_l,\boldsymbol{j}_t$)

$$A(x,t)=\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}A(k,\omega)e^{ikx-i\omega t} \tag{3.5}$$

we calculate the coefficient determinant of Eqs. (3.1)–(3.4) in leading (hydrodynamic) order of $\boldsymbol{k}$ and ω [17]. The result is

$$\Delta_H(k,\omega)=(-i\omega+D_t k^2)(-i\omega+D_T k^2)(\omega^2-c_s^2k^2+D_s i\omega k^2). \tag{3.6}$$

The coefficients are defined as

$$D_t=\frac{\overline{\eta}}{\rho},\quad D_T=\frac{\kappa_T}{\rho C_P}, \tag{3.7}$$

$$c_s^2=\left(\frac{\partial P}{\partial\rho}\right)_\sigma,\quad D_s=\frac{1}{\rho}\left(\zeta+\frac{4}{3}\overline{\eta}\right)+\frac{\kappa_T}{\rho}\left(\frac{1}{C_V}-\frac{1}{C_P}\right) \tag{3.8}$$

with the isochoric specific heat $C_V=T(\partial\sigma/\partial T)_\rho$ and the isobaric specific heat $C_P=T(\partial\sigma/\partial T)_P$. Equation (3.6) contains two diffusion modes and one sound mode. The shear diffusion coefficient D_t and the thermal diffusion coefficient D_T describe the transverse shear mode and heat diffusion in the liquid. The sound mode is described by the sound velocity c_s and sound diffusion coefficient D_s, which determines the sound attenuation

$$\alpha=\frac{\omega^2}{2c_s^3}D_s \tag{3.9}$$

measured in experiments. The contributions to the dynamic equations that are linear in the densities determine the Gaussian part (C10) of the dynamic functional (C9) defined in Appendix C and therefore the lowest order of the dynamic two point vertex functions. In the noncritical background the model dynamic equations describe hydrodynamics, therefore the Gaussian part has to be consistent with the Gaussian functional following from the hydrodynamic equations. Thus we get a connection between background hydrodynamics and the lowest order of the dynamic vertex functions. Approaching the critical temperature, fluctuations arise that lead to critical contributions to the thermodynamic derivatives and in the Onsager coefficients of the hydrodynamic equations but do not influence the structure of the equations. Within the model the effects of fluctuation are calculated in a perturbation expansion of the full dynamic functional and are contained in the perturbation contributions to the dynamic two point vertex functions (C14). Replacing the lowest order vertex functions by the full ones in the critical region, we keep the same formal relations between the hydrodynamic transport coefficients and the vertex functions as in the background. From perturbation expansion one can see that the dynamic vertex functions introduced in Eq. (C16) have the structure

$$[\mathring{\Gamma}_{\alpha\tilde{\beta}}]=\begin{pmatrix} -i\omega+k^2\mathring{f}_{\phi\tilde{\phi}} & k^2\mathring{f}_{\phi\tilde{q}} & k\mathring{g}_{\phi\tilde{l}} & 0\\ k^2\mathring{f}_{q\tilde{\phi}} & -i\omega+k^2\mathring{f}_{q\tilde{q}} & k\mathring{g}_{q\tilde{l}} & 0\\ k\mathring{g}_{l\tilde{\phi}} & k\mathring{g}_{l\tilde{q}} & -i\omega+k^2\mathring{f}_{l\tilde{l}} & 0\\ 0 & 0 & 0 & -i\omega+k^2\mathring{f}_{t\tilde{t}}\end{pmatrix}. \tag{3.10}$$

The complex functions $\mathring{f}_{\alpha\tilde{\beta}}$ and $\mathring{g}_{\alpha\tilde{\beta}}$ depend on the static and dynamic model parameters as well as on k and ω. In the hydrodynamic limit we take $k=0$ and keep the frequency finite. Thus the vertex functions on the right hand side of Eq. (3.10) are defined as the derivatives

$$\mathring{f}_{\alpha\tilde{\beta}}(\mathring{\tau},\mathring{\Omega},\{\mathring{\underline{\underline{\Xi}}}\})=\frac{\partial}{\partial k^2}\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\mathring{\tau},\mathring{\Omega},\{\mathring{\underline{\underline{\Xi}}}\})|_{k=0}, \tag{3.11}$$

$$\mathring{g}_{\alpha\tilde{\beta}}(\mathring{\tau},\mathring{\Omega},\{\mathring{\underline{\underline{\Xi}}}\})=\frac{\partial}{\partial k}\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\mathring{\tau},\mathring{\Omega},\{\mathring{\underline{\underline{\Xi}}}\})|_{k=0}, \tag{3.12}$$

where we have introduced the frequency parameter

$$\mathring{\Omega}=\frac{\omega}{2\mathring{\Gamma}}. \tag{3.13}$$

$\{\mathring{\Xi}\}$ represents the set of the static and dynamic model parameters $\{\mathring{\gamma}_q, \mathring{\tilde{u}}, \mathring{\Gamma}, \mathring{L}_\phi, \mathring{\lambda}, \mathring{\lambda}_l, \mathring{\lambda}_t, \mathring{g}, \mathring{g}_l\}$. Calculating the determinant of Eq. (3.10) in leading order we get

$$\Delta_{\rm th} = (-i\omega + \mathcal{D}_t k^2)(-i\omega + \mathcal{D}_T k^2)(\omega^2 - \mathcal{C}_s^2 k^2 + \mathcal{D}_s i\omega k^2). \tag{3.14}$$

The coefficients are defined by the equations

$$\mathcal{C}_s^2 = -(\mathring{g}_{q\tilde{l}}\mathring{g}_{l\tilde{q}} + \mathring{g}_{\phi\tilde{l}}\mathring{g}_{l\tilde{\phi}}), \tag{3.15}$$

$$\mathcal{D}_t\mathcal{D}_T = \frac{\mathring{f}_{t\tilde{t}}}{\mathcal{C}_s^2}[\mathring{f}_{\phi\tilde{\phi}}\mathring{g}_{q\tilde{l}}\mathring{g}_{l\tilde{q}} + \mathring{f}_{q\tilde{q}}\mathring{g}_{\phi\tilde{l}}\mathring{g}_{l\tilde{\phi}} - \mathring{f}_{\phi\tilde{q}}\mathring{g}_{q\tilde{l}}\mathring{g}_{l\tilde{\phi}} - \mathring{f}_{q\tilde{\phi}}\mathring{g}_{l\tilde{q}}\mathring{g}_{\phi\tilde{l}}], \tag{3.16}$$

$$\mathcal{D}_s = \mathring{f}_{\phi\tilde{\phi}} + \mathring{f}_{q\tilde{q}} + \mathring{f}_{l\tilde{l}} - \frac{\mathcal{D}_t\mathcal{D}_T}{\mathring{f}_{t\tilde{t}}}, \tag{3.17}$$

$$\mathcal{D}_t + \mathcal{D}_T = \mathring{f}_{t\tilde{t}} + \frac{\mathcal{D}_t\mathcal{D}_T}{\mathring{f}_{t\tilde{t}}}. \tag{3.18}$$

Although equation (3.14) looks like Eq. (3.6) we have to note that the coefficients in Eq. (3.14) are complex quantities. The thermal diffusion coefficient D_T and the shear diffusion coefficient D_t are simply determined by the real parts of the corresponding complex coefficients

$$D_T = \mathrm{Re}[\mathcal{D}_T], \quad D_t = \mathrm{Re}[\mathcal{D}_t]. \tag{3.19}$$

Expressions for the sound velocity c_s and the sound diffusion D_s may be obtained from the dispersion relation. From Eq. (3.6) the dispersion relation $\omega^2 = (c_s^2 - i\omega D_s)k^2$ follows while the corresponding expression from Eq. (3.14) is $\omega^2 = (\mathcal{C}_s^2 - i\omega\mathcal{D}_s)k^2$. Comparing real and imaginary parts allows one to identify the frequency-dependent sound velocity and sound diffusion:

$$c_s^2(t,\omega) = \mathrm{Re}[\mathcal{C}_s^2(t,\omega) - i\omega\mathcal{D}_s(t,\omega)],$$

$$D_s(t,\omega) = -\frac{1}{\omega}\mathrm{Im}[\mathcal{C}_s^2(t,\omega) - i\omega\mathcal{D}_s(t,\omega)]. \tag{3.20}$$

The frictional coefficient β [8,9] measured in an oscillating disk experiment is related to the complex shear viscosity

$$\overline{\eta} = \rho\mathcal{D}_t \tag{3.21}$$

by

$$\beta(t,\omega) \approx [\mathrm{Re}\,\overline{\eta(t,\omega)}] + \mathrm{Im}[\overline{\eta(t,\omega)}]. \tag{3.22}$$

Analogous to the λ transition in ^{4}He [6,7] the model only becomes renormalizable when the slow heat and shear modes are separated from the fast sound mode. This means that the dynamic model has to be considered in the limit $\mathring{c}\to\infty$. The structure of the perturbation theory gets simpler and one can see that no contributions arise to $\mathring{g}_{l\tilde{\phi}}$. In this limit also the dynamic functional reduces to model H of Siggia, Halperin, and Hohenberg [1] when the secondary density q_0 and the longitudinal momentum density $\boldsymbol{j}_l$ are eliminated by integration. Thus in the limit $\mathring{c}\to\infty$ we have $\mathring{g}_{l\tilde{\phi}} = 0$. Equations (3.15)–(3.18) simplify to

$$\mathcal{C}_s^2 = -\mathring{g}_{q\tilde{l}}\mathring{g}_{l\tilde{q}}, \quad \mathcal{D}_s = \mathring{f}_{q\tilde{q}} + \mathring{f}_{l\tilde{l}} + \frac{\mathring{f}_{q\tilde{\phi}}\mathring{g}_{\phi\tilde{l}}}{\mathring{g}_{q\tilde{l}}}, \tag{3.23}$$

$$\mathcal{D}_t = \mathring{f}_{t\tilde{t}}, \quad \mathcal{D}_T = \mathring{f}_{\phi\tilde{\phi}} - \frac{\mathring{f}_{q\tilde{\phi}}\mathring{g}_{\phi\tilde{l}}}{\mathring{g}_{q\tilde{l}}}. \tag{3.24}$$

From Eqs. (C11) and (C12) and the structure of the perturbation theory follows that $\mathring{g}_{q\tilde{l}}$ is proportional to $\mathring{c}$ while from $\mathring{g}_{\phi\tilde{l}}$ no factor $\mathring{c}$ can be extracted. Therefore the last term in $\mathcal{D}_s$ and $\mathcal{D}_T$ is proportional to $1/\mathring{c}$ and may be neglected in the limit $\mathring{c}\to\infty$. Inserting Eqs. (3.23) and (3.24) into Eqs. (3.19) and (3.20) the hydrodynamic transport coefficients are

$$c_s^2 = -\mathrm{Re}[\mathring{g}_{q\tilde{l}}\mathring{g}_{l\tilde{q}} + i\omega(\mathring{f}_{q\tilde{q}} + \mathring{f}_{l\tilde{l}})], \tag{3.25}$$

$$D_s = \frac{1}{\omega}\mathrm{Im}[\mathring{g}_{q\tilde{l}}\mathring{g}_{l\tilde{q}} + i\omega(\mathring{f}_{q\tilde{q}} + \mathring{f}_{l\tilde{l}})], \tag{3.26}$$

$$D_t = \mathrm{Re}[\mathring{f}_{t\tilde{t}}], \quad D_T = \mathrm{Re}[\mathring{f}_{\phi\tilde{\phi}}]. \tag{3.27}$$

From perturbation theory one can see that in the case of finite frequencies we may extract functions $\mathring{\Gamma}_{\alpha\beta}$ from the k derivatives of the vertex functions

$$[\mathring{f}_{\alpha\tilde{\beta}}] = \begin{pmatrix} \mathring{f}_{\phi\tilde{\phi}} & \mathring{f}_{\phi\tilde{q}} & \mathring{g}_{\phi\tilde{l}} & 0 \\ \mathring{f}_{q\tilde{\phi}} & \mathring{f}_{q\tilde{q}} & \mathring{g}_{q\tilde{l}} & 0 \\ 0 & \mathring{g}_{l\tilde{q}} & \mathring{f}_{l\tilde{l}} & 0 \\ 0 & 0 & 0 & \mathring{f}_{t\tilde{t}} \end{pmatrix}, \tag{3.28}$$

which have the property that they reduce at vanishing frequencies to the static vertex functions $\mathring{\Gamma}^{(s)}_{\alpha\beta}$ calculated within the extended model (2.10). Particularly we have

$$\lim_{\omega\to 0}\mathring{\Gamma}_{\alpha\beta}(\mathring{\tau},\mathring{\Omega},\{\mathring{\Xi}\}) = \mathring{\Gamma}^{(s)}_{\alpha\beta}(\mathring{\tau},\mathring{\gamma}_q,\mathring{\tilde{u}}). \tag{3.29}$$

Thus the matrix (3.28) may be written as the product

$$[\mathring{f}_{\alpha\tilde{\beta}}](\mathring{\tau},\mathring{\Omega},\{\mathring{\Xi}\}) = [\mathring{\Gamma}_{\alpha\beta}](\mathring{\tau},\mathring{\Omega},\{\mathring{\Xi}\})[\mathring{f}^{(d)}_{\alpha\tilde{\beta}}](\mathring{\tau},\mathring{\Omega},\{\mathring{\Xi}\}). \tag{3.30}$$

At zero frequency Eq. (3.30) coincides with results that have been obtained for liquids and liquid mixtures calculated within model H and model H′ [10,18,4]. The matrices on the right hand side of Eq. (3.30) are defined as

$$[\mathring{\Gamma}_{\alpha\beta}]=\begin{pmatrix} \mathring{\Gamma}_{\phi\phi} & 0 & 0 & 0 \\ 0 & \mathring{\Gamma}_{qq} & 0 & 0 \\ 0 & 0 & \Gamma_{ll} & 0 \\ 0 & 0 & 0 & \Gamma_{tt} \end{pmatrix},$$

$$[\mathring{f}^{(d)}_{\alpha\tilde{\beta}}]=\begin{pmatrix} \mathring{f}^{(d)}_{\phi\tilde{\phi}} & \mathring{f}^{(d)}_{\phi\tilde{q}} & \mathring{g}^{(d)}_{\phi\tilde{l}} & 0 \\ \mathring{f}^{(d)}_{q\tilde{\phi}} & \mathring{f}^{(d)}_{q\tilde{q}} & \mathring{g}^{(d)}_{q\tilde{l}} & 0 \\ 0 & \mathring{g}^{(d)}_{l\tilde{q}} & \mathring{f}^{(d)}_{l\tilde{l}} & 0 \\ 0 & 0 & 0 & \mathring{f}^{(d)}_{t\tilde{t}} \end{pmatrix}. \tag{3.31}$$

Γ_{ll} and Γ_{tt} appearing in Eq. (3.31) are also at finite frequencies equal to the constant $\Gamma_{ll}=\Gamma_{tt}=a_j$, thus we do not need any distinction between static functions and frequency-dependent functions.

In order to introduce the correlation length ξ as the temperature scale in all vertex functions, we replace in a first step the parameters $\mathring{\tau}$ and $\mathring{\tilde{u}}$ of the extended model (2.10) by the corresponding parameters $\mathring{r}$ and $\mathring{u}$ of the ϕ^4 model (2.18) using Eqs. (2.19) and (2.22). This allows the calculation of the vertex functions by ϵ expansion as well as in $d=3$ [19]. This step leads at finite frequency to vertex functions $\mathring{\Gamma}_{\alpha\beta}(\mathring{r},\mathring{\Omega},\{\mathring{\Xi}\})$, $\mathring{f}^{(d)}_{\alpha\tilde{\beta}}(\mathring{r},\mathring{\Omega},\{\mathring{\Xi}\})$, and $\mathring{g}^{(d)}_{\alpha\tilde{\beta}}(\mathring{r},\mathring{\Omega},\{\mathring{\Xi}\})$, where $\{\Xi\}$ represents now the set $\{\mathring{\gamma}_q,\mathring{u},\mathring{\Gamma},\mathring{L}_\phi,\mathring{\lambda},\mathring{\lambda}_l,\mathring{\lambda}_t,\mathring{g},\mathring{g}_l\}$. Analogously the static vertex function $\mathring{\Gamma}^{(s)}_{qq}(\mathring{\tau},\mathring{\gamma}_q,\mathring{\tilde{u}})$ of the secondary density q_0 change to $\mathring{\Gamma}^{(s)}_{qq}(\mathring{r},\mathring{\gamma}_q,\mathring{u})$. The static order parameter vertex function $\mathring{\Gamma}^{(s)}_{\phi\phi}(\mathring{\tau},\mathring{\gamma}_q,\mathring{\tilde{u}})$ calculated within the extended model is equal to the corresponding function calculated within the ϕ^4 model. Therefore expressing this function by the ϕ^4-model parameters the explicit $\mathring{\gamma}_q$ dependence drops out and we get the ϕ^4-model order parameter vertex function $\mathring{\Gamma}^{(s)}_{\phi\phi}(\mathring{r},\mathring{u})$. The vertex functions obtained so far contain dimensional poles not only at $d=4$ but also at $d=3$ when the cut off wave number is shifted to infinity. In the second step we have to remove the $d=3$ poles from the vertex functions, which is also necessary to obtain a defined expression for the correlation length at $d=3$. This can be done by inserting explicitly the shift of the critical temperature characterized by a critical value $\mathring{r}_c$ into the vertex functions [19]. The critical value $\mathring{r}_c$ is defined by the condition $\mathring{\Gamma}^{(s)}_{\phi\phi}(\mathring{r}_c,\mathring{u})=0$ from which $\mathring{r}_c(\mathring{u})$ may be calculated by inversion in every order of perturbation expansion. All vertex function may be expressed as a function of $\mathring{r}-\mathring{r}_c$ by inserting $\mathring{r}=\mathring{r}-\mathring{r}_c+\mathring{r}_c$. The correlation length is defined by [13]

$$\xi^2(\mathring{r}-\mathring{r}_c,\mathring{u})=\left.\frac{\partial \ln \mathring{\Gamma}^{(s)}_{\phi\phi}(k,\mathring{r}-\mathring{r}_c,\mathring{u})}{\partial k^2}\right|_{k=0}, \tag{3.32}$$

which is a finite expression at $d=3$ [19]. Inserting $\mathring{r}-\mathring{r}_c=F(\xi^{-2},\mathring{u})$ into the vertex functions, where F may be calculated by inversion of Eq. (3.32), the temperature scale is now expressed by the inverse correlation length getting functions $\mathring{\Gamma}_{\alpha\beta}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})$, $\mathring{f}^{(d)}_{\alpha\tilde{\beta}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})$ and $\mathring{g}^{(d)}_{\alpha\tilde{\beta}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})$.

To make the procedure mentioned above more clear, we will apply it in the following to the vertex functions calculated in one loop order showing the main steps for the static vertex functions. The static vertex functions calculated within the extended model (2.10) are

$$\mathring{\Gamma}^{(s)}_{\phi\phi}(\mathring{\tau},\mathring{\gamma}_q,\mathring{\tilde{u}})=\mathring{\tau}\left(\frac{\mathring{\tilde{u}}}{2}-\frac{\mathring{\gamma}_q^2}{a_q}\right)I_1(\mathring{\tau}), \tag{3.33}$$

$$\mathring{\Gamma}^{(s)}_{qq}(\mathring{r},\mathring{\gamma}_q,\mathring{\tilde{u}})=a_q\left(1-\frac{\mathring{\gamma}_q^2}{2a_q}I_2(\mathring{\tau})\right), \tag{3.34}$$

where we have defined

$$I_m(\mathring{\tau})=\int_{k'}\frac{1}{(\mathring{\tau}+k'^2)^m}. \tag{3.35}$$

Note that the integral I_1 appearing in the order parameter vertex function contains a dimensional pole at $d=3$ for an infinite cutoff wave number. In the above equations we may insert the one loop expression of Eq. (2.22)

$$\mathring{r}=\mathring{\tau}+\frac{\mathring{\gamma}_q^2}{2a_q}I_1(\mathring{r}) \tag{3.36}$$

and also Eq. (2.19) to change the extended model parameters $\mathring{\tau}$ and $\mathring{\tilde{u}}$ to the parameters $\mathring{r}$ and $\mathring{u}$ of the ϕ^4 model (2.18). The result is

$$\mathring{\Gamma}^{(s)}_{\phi\phi}(\mathring{r},\mathring{u})=\mathring{r}+\frac{\mathring{u}}{2}I_1(\mathring{r}), \tag{3.37}$$

$$\mathring{\Gamma}^{(s)}_{qq}(\mathring{r},\mathring{\gamma}_q,\mathring{u})=a_q\left(1-\frac{\mathring{\gamma}_q^2}{2a_q}I_2(\mathring{r})\right). \tag{3.38}$$

The order parameter vertex function (3.37) is now equal to the expression obtained by a calculation within the ϕ^4 model [13]. The critical value $\mathring{r}_c$ is in one loop order defined by the equation

$$\mathring{r}_c+\frac{\mathring{u}}{2}I_1(\mathring{r}_c)=0. \tag{3.39}$$

Inserting the zero loop solution $\mathring{r}_c=0$ into the first order term of (3.39) the critical value in one loop order is

$$\mathring{r}_c=-\frac{\mathring{u}}{2}I_1(0). \tag{3.40}$$

Replacing $\mathring{r}$ by $\mathring{r}-\mathring{r}_c+\mathring{r}_c$ in Eqs. (3.37) and (3.38) we get

$$\mathring{\Gamma}^{(s)}_{\phi\phi}(\mathring{r}-\mathring{r}_c,\mathring{u})=\mathring{r}-\mathring{r}_c+\frac{\mathring{u}}{2}[I_1(\mathring{r}-\mathring{r}_c)-I_1(0)], \tag{3.41}$$

$$\mathring{\Gamma}^{(s)}_{qq}(\mathring{r}-\mathring{r}_c,\mathring{\gamma}_q,\mathring{u})=a_q\left(1-\frac{\mathring{\gamma}_q^2}{2a_q}I_2(\mathring{r}-\mathring{r}_c)\right). \tag{3.42}$$

The order parameter vertex function at finite wave number in one loop order is simply given by $\mathring{\Gamma}^{(s)}_{\phi\phi}(\mathring{r}-\mathring{r}_c,\mathring{u})+k^2$. Inserting in Eq. (3.32) the resulting expression for the correlation length is

$$\xi^{-2}(\mathring{r}-\mathring{r}_c,\mathring{u})=\mathring{r}-\mathring{r}_c+\frac{\mathring{u}}{2}[I_1(\mathring{r}-\mathring{r}_c)-I_1(0)]. \tag{3.43}$$

In contrast to $I_1(\mathring{r})$ the integral

$$I_1(\mathring{r}-\mathring{r}_c)-I_1(0)=-\int_{k'}\frac{\mathring{r}-\mathring{r}_c}{k'^2(\mathring{r}-\mathring{r}_c+k'^2)} \tag{3.44}$$

contains no poles at $d=3$ and Eq. (3.43) is a well defined equation at three dimensions. All $d=3$ poles have been absorbed in the critical value $\mathring{r}_c$. The correlation function may be introduced in the vertex functions (3.41) and (3.42) by inserting the inverse of Eq. (3.43). The resulting one loop functions are

$$\mathring{\Gamma}^{(s)}_{\phi\phi}(\xi^{-2},\mathring{u})=\xi^{-2},$$

$$\mathring{\Gamma}^{(s)}_{qq}(\xi^{-2},\mathring{\gamma}_q,\mathring{u})=a_q\left[1-\frac{\mathring{\gamma}_q^2}{2a_q}I_2(\xi^{-2})\right]. \tag{3.45}$$

The above procedure may be performed in every order of perturbation expansion removing all poles in three dimensions in the model functions [19]. Analogous to Eqs. (3.33)–(3.45) the correlation length also will be introduced in the vertex functions calculated at finite frequency. $\mathring{\Gamma}_{\phi\phi}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})$ does not depend explicitly on the frequency in one loop order. We get

$$\mathring{\Gamma}_{\phi\phi}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{\Gamma}^{(s)}_{\phi\phi}(\xi^{-2},\mathring{u}). \tag{3.46}$$

The one loop expression of $\mathring{\Gamma}_{qq}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})$ is given by

$$\mathring{\Gamma}_{qq}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})$$

$$=a_q\left(1-\frac{\mathring{\gamma}_q^2}{2a_q}\int_{k'}\frac{k'^2}{(\xi^{-2}+k'^2)[-i\mathring{\Omega}+k'^2(\xi^{-2}+k'^2)]}\right) \tag{3.47}$$

from which one can immediately see that it reduces to the corresponding static function in Eq. (3.45) for $\omega=0$. Analogous in model H [1] the time scale ratios

$$\mathring{w}_t=\frac{\mathring{\Gamma}}{a_j\mathring{\lambda}_t},\quad \mathring{f}_t=\frac{\mathring{g}}{\sqrt{\mathring{\Gamma}\mathring{\lambda}_t}} \tag{3.48}$$

will be introduced. In the present model additional ratios may be introduced corresponding to the longitudinal momentum density

$$\mathring{w}_l=\frac{\mathring{\Gamma}}{a_j\mathring{\lambda}_l},\quad \mathring{f}_l=\frac{\mathring{g}}{\sqrt{\mathring{\Gamma}\mathring{\lambda}_l}}. \tag{3.49}$$

The cutoff dimensions $\mathring{w}_t\sim\Lambda^{-2}$ and $\mathring{w}_l\sim\Lambda^{-2}$ are negative, which means that the parameters are irrelevant for the critical theory. The renormalization in the following section will be performed at vanishing irrelevant parameters [20]. Therefore it is sufficient to calculate the vertex functions in Eqs. (3.25)–(3.27) at $\mathring{w}_t=\mathring{w}_l=0$. Introducing the correlation length (3.43) we get for the purely dynamic parts of the vertex functions in one loop order

$$\mathring{\Gamma}^{(d)}_{\phi\tilde{\phi}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{\Gamma}\left\{1-\frac{\mathring{f}_t^2}{\xi^{-2}}\int_{k'}\frac{\sin^2\theta}{\xi^{-2}+k'^2}\right\}, \tag{3.50}$$

$$\mathring{\Gamma}^{(d)}_{t\tilde{t}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{\lambda}_t\left\{1+\frac{\mathring{f}_t^2}{6}\left[\int_{k'}\frac{k'^2\sin^2\theta}{(\xi^{-2}+k'^2)[-i\mathring{\Omega}+k'^2(\xi^{-2}+k'^2)]}-2\int_{k'}\frac{k'^4\sin^2\theta\cos^2\theta}{[-i\mathring{\Omega}+k'^2(\xi^{-2}+k'^2)]^2}\left(1+\frac{k'^2}{\xi^{-2}+k'^2}\right)\right]\right\}, \tag{3.51}$$

$$\mathring{\Gamma}^{(d)}_{l\tilde{l}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{\lambda}_l\left\{1+\frac{\mathring{f}_l^2}{6}\left[\int_{k'}\frac{k'^2}{(\xi^{-2}+k'^2)[-i\mathring{\Omega}+k'^2(\xi^{-2}+k'^2)]}+\int_{k'}\frac{(\xi^{-2}-k'^2)k'^2\cos^2\theta}{[-i\mathring{\Omega}+k'^2(\xi^{-2}+k'^2)]^2}\left(1+\frac{k'^2}{\xi^{-2}+k'^2}\right)\right]\right\}, \tag{3.52}$$

The second integrals within the braces in Eqs. (3.51) and (3.52) are obtained after performing the derivatives of the one loop contribution according to the definition in Eq. (3.11).

$$\mathring{\tilde{f}}^{(d)}_{q\tilde{q}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{\lambda}, \tag{3.53}$$

$$\mathring{\tilde{f}}^{(d)}_{q\tilde{\phi}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{\tilde{f}}^{(d)}_{\phi\tilde{q}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{L}_{\phi}, \tag{3.54}$$

$$\mathring{g}^{(d)}_{q\tilde{l}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=\mathring{g}^{(d)}_{l\tilde{q}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=i\mathring{c}. \tag{3.55}$$

Note that in Eq. (3.50) no frequency dependence appears explicitly at $\mathring{w}_t=0$ in one loop order and therefore it is equal to the expression obtained in the reduced model H or mixture model H′ for $\mathring{w}_3=0$, respectively (compare [10]). The dynamic vertex function of the transverse mode (3.51) reduces in the limit $\omega\to 0$ to the corresponding model H function [1]. Inserting Eqs. (3.30), (3.46), (3.47), and (3.50)–(3.55) into Eqs. (3.25)–(3.27), the expressions for the hydrodynamic transport coefficient are expressed by the unrenormalized model parameters.

IV. RENORMALIZATION

A. Statics

The model will be treated within the field-theoretic renormalization group theory. Using the minimal subtraction scheme [13] dimensional singularities at space dimension $d=4$ in the vertex functions will be absorbed into Z factors.

The longitudinal and transverse momentum density need no renormalization because these densities only enter in quadratic order in the extended model Hamiltonian. The renormalization of Eq. (2.10) follows along the same lines as in model F for the superfluid transition in ^{4}He [20,14], which is well known and therefore only will be sketched briefly in the following. Calculation details are comprehensively shown in [14] (here we have the case of a one component order parameter and the specific heat is diverging). The order parameter and the secondary field are renormalized by

$$\phi_0=Z_\phi^{1/2}\phi,\quad q_0=Z_q^{1/2}q. \tag{4.1}$$

From Eq. (2.21) it follows immediately that Z_q is determined by the singularities of the ϕ^2-ϕ^2 correlation function, which is proportional to the specific heat

$$Z_q^{-1}=1+\frac{\gamma_q^2}{a_q}A(u). \tag{4.2}$$

$A(u)$ contains the singularities of the specific heat calculated within the ϕ^4 model and is obtained by an additive renormalization of the ϕ^2-ϕ^2 correlation function. u is the renormalized fourth order coupling of the ϕ^4 model (2.18) in which renormalized parameters will be introduced by

$$\mathring{r}-\mathring{r}_c=Z_\phi^{-1}Z_r r,\quad \mathring{u}=\kappa^{\epsilon}Z_\phi^{-2}Z_u u A_d^{-1}. \tag{4.3}$$

κ is a reference wave number that will be specified later and $\epsilon=d-4$. The factor $A_d=\Gamma(3-d/2)/[2^{d-2}\pi^{d/2}(d-2)]$ has been chosen to obtain a minimal number of perturbation contributions in an ϵ expansion of the specific heat [14]. In the extended model (2.10) we have additionally

$$\mathring{\gamma}_q=\kappa^{\epsilon/2}Z_\phi^{-1}Z_q^{-1/2}Z_\gamma\gamma_q A_d^{-1/2}. \tag{4.4}$$

Since Z_γ in Eq. (4.4) may be replaced by Z factors of the ϕ^4 model via the relation

$$Z_\gamma=Z_q Z_r \tag{4.5}$$

the critical singularities are completely the same as in the ϕ^4 model (2.18), no new singularities appear in the extended model (2.10). From the Z factors we define the ζ functions

$$\zeta_i=\left(\kappa\frac{\partial \ln Z_i^{-1}}{\partial\kappa}\right)_0,\quad i=\phi,q,r,u. \tag{4.6}$$

The index 0 in Eq. (4.6) indicates that the derivative is taken at fixed unrenormalized parameters. The renormalization of the static parameters is described by the flow equations. The temperature dependence of these parameters is then determined from the l dependence via a matching condition (see below)

$$l\frac{dr}{dl}=r[\zeta_r(u)-\zeta_\phi(u)], \tag{4.7}$$

$$l\frac{d\gamma_q}{dl}=\gamma_q\left[-\frac{\epsilon}{2}-\zeta_\phi(u)+\frac{1}{2}\zeta_q(\gamma_q,u)+\zeta_r(u)\right], \tag{4.8}$$

$$l\frac{du}{dl}=u[-\epsilon-2\zeta_\phi(u)+\zeta_u(u)]. \tag{4.9}$$

From Eq. (4.9) one obtains the Heisenberg fixed point $u_H^\star$ as the stable one [21,22]. For a one component order parameter a finite fixed point $\gamma_q^\star$ follows from Eq. (4.8). The fixed point values $\zeta_i^\star$ of the ζ functions are related to the critical exponents by

$$\zeta_\phi^\star=-\eta,\quad \zeta_q^\star=\frac{\alpha}{\nu},\quad \zeta_r^\star-\zeta_\phi^\star=2-\frac{1}{\nu}. \tag{4.10}$$

The fixed point value of the coupling γ_q is related to the fixed point value of $\zeta_q^\star$ [23,24]. In one loop order the relation reads $\gamma_q^{2\star}/a_q=2\zeta_q^\star$.

The renormalization of the static vertex functions reads

$$\Gamma^{(s)}_{\phi\phi}(\xi^{-2},u,\kappa)=Z_\phi\mathring{\Gamma}^{(s)}_{\phi\phi}(\xi^{-2},\mathring{u}), \tag{4.11}$$

$$\Gamma^{(s)}_{qq}(\xi^{-2},\gamma_q,u,\kappa)=Z_q\mathring{\Gamma}^{(s)}_{qq}(\xi^{-2},\mathring{\gamma}_q,\mathring{u}). \tag{4.12}$$

From the corresponding standard renormalization group equations in field theory [13] follow the solutions

$$\Gamma^{(s)}_{\phi\phi}(\xi^{-2},u,\kappa)=(\kappa l)^2 e^{\int_1^l (dx/x)\zeta_\phi}\hat{\Gamma}^{(s)}_{\phi\phi}\left(\frac{\xi^{-2}(t)}{(\kappa l)^2},u(l)\right), \tag{4.13}$$

$$\Gamma_{qq}^{(s)}(\xi^{-2},\gamma_q,u,\kappa)=e^{\int_1^l (dx/x)\zeta_q}\hat{\Gamma}_{qq}^{(s)}\left(\frac{\xi^{-2}(t)}{(\kappa l)^2},\gamma_q(l),u(l)\right). \tag{4.14}$$

$\hat{\Gamma}_{\phi\phi}^{(s)}$ and $\hat{\Gamma}_{qq}^{(s)}$ are the amplitude functions, which are determined by the corresponding two point vertex functions calculated in a loop expansion. In order to guarantee finite amplitude functions in the critical limit $t\to 0$ usually one chooses [19]

$$\frac{\xi^{-2}(t)}{(\kappa l)^2}=1, \tag{4.15}$$

which defines the relation between reduced temperature and flow parameter. All terms containing logarithms of Eq. (4.15) vanish in the amplitude functions. The critical singularities are then completely collected in the prefactors of Eqs. (4.13) and (4.14), while the amplitude functions stay smooth and finite. Specifying the wave number $\kappa=\xi_0^{-1}$ in Eq. (4.15), the relation between reduced temperature and flow parameter is written as

$$l=\xi_0\xi^{-1}(t). \tag{4.16}$$

Thus the static vertex functions are related to the renormalized parameters by

$$\mathring{\Gamma}_{\phi\phi}^{(s)}(\xi^{-2},\mathring{u})=(\kappa l)^2 Z_\phi^{-1}e^{\int_1^l (dx/x)\zeta_\phi}\hat{\Gamma}_{\phi\phi}^{(s)}(u(l)), \tag{4.17}$$

$$\mathring{\Gamma}_{qq}^{(s)}(\xi^{-2},\mathring{\gamma}_q,\mathring{u})=Z_q^{-1}e^{\int_1^l (dx/x)\zeta_q}\hat{\Gamma}_{qq}^{(s)}(\gamma_q(l),u(l)). \tag{4.18}$$

In one loop order we simply have $\hat{\Gamma}_{\phi\phi}^{(s)}(u(l))=1$. Equations (2.17) and (2.21) give rise to the exact relation

$$\hat{\Gamma}_{qq}^{(s)}(\gamma_q(l),u(l))=\frac{a_q}{1+[\gamma_q^2(l)/a_q]F_+^{(s)}(u(l))} \tag{4.19}$$

where $F_+^{(s)}(u(l))$ is the amplitude of the $\langle\phi^2\phi^2\rangle_c$ correlation function calculated within the ϕ^4 model [14].

B. Dynamics

The conjugated fields $\tilde{\phi}_0$, $\tilde{q}_0$, $\tilde{\boldsymbol{j}}_{l0}$, and $\tilde{\boldsymbol{j}}_{t0}$ introduced in the Appendixes are renormalized analogous to Eq. (4.1).

$$\tilde{\phi}_0=Z_{\tilde{\phi}}^{1/2}\tilde{\phi},\quad \tilde{q}_0=Z_{\tilde{q}}^{1/2}\tilde{q}, \tag{4.20}$$

$$\tilde{\boldsymbol{j}}_{l0}=Z_{\tilde{l}}^{1/2}\tilde{\boldsymbol{j}}_l,\quad \tilde{\boldsymbol{j}}_{t0}=Z_{\tilde{t}}^{1/2}\tilde{\boldsymbol{j}}_t. \tag{4.21}$$

All hydrodynamic denisities are conserved densities. The frequency terms of the dynamic vertex functions $(\partial\mathring{\Gamma}_{\alpha\tilde{\beta}}/\partial\omega)|_{\omega=0}$ do not contain ϵ poles [20]. As a consequence we get

$$Z_{\tilde{\phi}}=Z_\phi^{-1},\quad Z_{\tilde{q}}=Z_q^{-1},\quad Z_{\tilde{l}}=1,\quad Z_{\tilde{t}}=1. \tag{4.22}$$

Using Ward identities, which are a consequence of the Galilean invariance of the equations of motion [20], one finds that the mode couplings need no Z factors. Thus the following renormalized couplings will be introduced:

$$\mathring{g}=\kappa^{1+\epsilon/2}gA_d^{-1/2},\quad \mathring{g}_l=\kappa^{2+\epsilon/2}g_lA_d^{-1/2}. \tag{4.23}$$

The Onsager coefficients renormalize as

$$\mathring{\Gamma}=Z_\Gamma\Gamma,\quad \mathring{L}_\phi=\kappa Z_{L_\phi}L_\phi,\quad \mathring{\lambda}=\kappa^2 Z_\lambda\lambda, \tag{4.24}$$

$$\mathring{\lambda}_l=\kappa^2 Z_{\lambda_l}\lambda_l,\quad \mathring{\lambda}_t=\kappa^2 Z_{\lambda_t}\lambda_t. \tag{4.25}$$

Using Eqs. (4.1), (4.20), and (4.22) the dynamic vertex functions defined in the Appendixes get the following renormalization factors:

$$\Gamma_{\phi\tilde{\phi}}=\mathring{\Gamma}_{\phi\tilde{\phi}},\quad \Gamma_{q\tilde{q}}=\mathring{\Gamma}_{q\tilde{q}},\quad \Gamma_{l\tilde{l}}=\mathring{\Gamma}_{l\tilde{l}},\quad \Gamma_{t\tilde{t}}=\mathring{\Gamma}_{t\tilde{t}}, \tag{4.26}$$

$$\Gamma_{\phi\tilde{q}}=Z_\phi^{1/2}Z_q^{-1/2}\mathring{\Gamma}_{\phi\tilde{q}},\quad \Gamma_{q\tilde{\phi}}=Z_\phi^{-1/2}Z_q^{1/2}\mathring{\Gamma}_{q\tilde{\phi}}, \tag{4.27}$$

$$\Gamma_{q\tilde{l}}=Z_q^{1/2}\mathring{\Gamma}_{q\tilde{l}},\quad \Gamma_{l\tilde{q}}=Z_q^{-1/2}\mathring{\Gamma}_{l\tilde{q}}a. \tag{4.28}$$

The vertex functions factorize into a static and dynamic part, as discussed in the previous section, as a consequence the Z factors defined in Eq. (4.24) also separate into a static and dynamic part. One gets

$$Z_\Gamma=Z_\phi Z_\Gamma^{(d)},\quad Z_\lambda=Z_qZ_\lambda^{(d)},\quad Z_{L_\phi}=Z_\phi^{1/2}Z_q^{1/2}Z_{L_\phi}^{(d)}. \tag{4.29}$$

The static vertex functions of the longitudinal and transverse momentum density are simply represented by the constant parameter a_j. Therefore Z_{λ_l} and Z_{λ_t} in Eq. (4.25) do not contain static ϵ poles. From Eqs. (3.30) and (4.26)–(4.28) it follows immediately that

$$f_{\phi\tilde{\phi}}^{(d)}=Z_\phi^{-1}\mathring{f}_{\phi\tilde{\phi}}^{(d)},\quad f_{q\tilde{q}}^{(d)}=Z_q^{-1}\mathring{f}_{q\tilde{q}}^{(d)}, \tag{4.30}$$

$$f_{\phi\tilde{q}}^{(d)}=Z_\phi^{-1/2}Z_q^{-1/2}\mathring{f}_{\phi\tilde{q}}^{(d)},\quad f_{q\tilde{\phi}}^{(d)}=Z_\phi^{-1/2}Z_q^{-1/2}\mathring{f}_{q\tilde{\phi}}^{(d)}, \tag{4.31}$$

$$g_{l\tilde{q}}^{(d)}=Z_q^{-1/2}\mathring{g}_{l\tilde{q}}^{(d)},\quad g_{q\tilde{l}}^{(d)}=Z_q^{-1/2}\mathring{g}_{q\tilde{l}}^{(d)}, \tag{4.32}$$

$$f_{l\tilde{l}}^{(d)}=\mathring{f}_{l\tilde{l}}^{(d)},\quad f_{t\tilde{t}}^{(d)}=\mathring{f}_{t\tilde{t}}^{(d)}. \tag{4.33}$$

In Eq. (3.53) one can see that the perturbation expansion does not contribute to $\mathring{g}_{l\tilde{q}}^{(d)}$, which means that no ϵ poles appear in this function. Therefore $\mathring{c}$ does not need an independent renormalization factor. The corresponding renormalized coefficient absorbs the Z factor in Eq. (4.32).

$$\mathring{c}=\kappa^3Z_q^{1/2}c. \tag{4.34}$$

Analogous to mixtures [10] no ϵ poles arise from perturbation theory to $f_{q\tilde{q}}^{(d)}$, $f_{\phi\tilde{q}}^{(d)}$, and $f_{\phi\tilde{q}}^{(d)}$ therefore we have $Z_{L_\phi}^{(d)}=1$ and $Z_\lambda^{(d)}=1$ in all orders of the loop expansion. In Eqs. (4.24) and (4.29) only the static renormalization factors remain in the above-mentioned Onsager coefficients. Γ, λ_l, and λ_t get nontrivial dynamic Z factors. We define ζ functions for $i=\Gamma,L_\phi,\lambda,\lambda_l,\lambda_t$ analogous to Eq. (4.6). Equation (4.29) implies a separation of ζ_Γ, ζ_{L_ϕ}, and ζ_λ into a static and dynamic part:

$$\zeta_\Gamma=\zeta_\Gamma^{(d)}+\zeta_\phi, \quad \zeta_\lambda=\zeta_q, \quad \zeta_{L_\phi}=\tfrac{1}{2}\zeta_q+\tfrac{1}{2}\zeta_\phi. \quad (4.35)$$

In the last two equations $\zeta_{L_\phi}^{(d)}=0$ and $\zeta_\lambda^{(d)}=0$ has been used. The critical temperature dependence of all parameters is determined by flow equations analogous to statics. The flow equations for the Onsager coefficients read

$$l\frac{d\Gamma}{dl}=\Gamma(\zeta_\Gamma^{(d)}+\zeta_\phi), \quad l\frac{d\lambda}{dl}=\lambda(-2+\zeta_q), \quad (4.36)$$

$$l\frac{dL_\phi}{dl}=L_\phi(-1+\tfrac{1}{2}\zeta_\phi+\tfrac{1}{2}\zeta_q), \quad (4.37)$$

$$l\frac{d\lambda_l}{dl}=\lambda_l(-2+\zeta_{\lambda_l}), \quad l\frac{d\lambda_t}{dl}=\lambda_t(-2+\zeta_{\lambda_t}). \quad (4.38)$$

Inserting the renormalized parameters (4.23), (4.24), and (4.25) into the definitions of the mode coupling parameters (3.48), (3.49) and into (4.34) we get the flow equations

$$l\frac{df_l}{dl}=-\frac{1}{2}f_l(\epsilon+\zeta_\Gamma^{(d)}+\zeta_{\lambda_l}+\zeta_\phi), \quad (4.39)$$

$$l\frac{df_t}{dl}=-\frac{1}{2}f_t(\epsilon+\zeta_\Gamma^{(d)}+\zeta_{\lambda_t}+\zeta_\phi), \quad (4.40)$$

$$l\frac{dc}{dl}=c(-3+\tfrac{1}{2}\zeta_q). \quad (4.41)$$

Equations (4.8), (4.9), and (4.36)–(4.41) completely determine the critical behavior of the static and dynamic model parameters.

At finite frequency no new ϵ poles appear in the vertex functions. The Z factors $Z_\Gamma^{(d)},Z_{\lambda_t}$ and therefore $\zeta_\Gamma^{(d)},\zeta_{\lambda_t}$ are identical to the corresponding quantities in model H. As a consequence the fixed point value of the mode coupling f_t is the same as in model H [1,20] ($f_t^\star=f_{\text{liquid}}^\star\neq 0$). From Eq. (4.40) we get at the fixed point

$$\zeta_{\lambda_t}^\star=-(\epsilon+\zeta_\Gamma^{(d)\star}+\zeta_\phi^\star) \quad (4.42)$$

and an analogous equation for $\zeta_{\lambda_l}^\star$ for the finite fixed point $f_l^\star$. Thus from Eqs. (4.39) and (4.40) the fixed point relation

$$\zeta_{\lambda_l}^\star=\zeta_{\lambda_t}^\star \quad (4.43)$$

is obtained.

V. CRITICAL BEHAVIOR OF THE TRANSPORT COEFFICIENTS AT FINITE FREQUENCIES

The transport coefficients are separated into static and dynamic contributions by inserting Eq. (3.30) in Eqs. (3.25)–(3.27). In order to obtain the critical behavior of the hydrodynamic transport coefficients we proceed quite analogously to the method used at the λ transition in ^{4}He [25] and ^{3}He-^{4}He mixtures [26]. We use the loop expansion only for the dynamic vertex functions $\mathring{f}_{\alpha\beta}^{(d)}$ and $\mathring{g}_{\alpha\beta}^{(d)}$ while the static vertex functions $\mathring{\Gamma}_{\alpha\beta}^{(s)}$ are replaced by the corresponding thermodynamic derivatives, which will be taken from experiments.

A. Shear viscosity and frictional coefficient

From the definition of the complex shear viscosity Eqs. (3.21) and (3.30) we have

$$\overline{\eta(t,\omega)}=\rho a_j \mathring{f}_{t\tilde{t}}^{(d)}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\}). \quad (5.1)$$

Inserting the solution of the renormalization group equation for the vertex function used in standard renormalzation group theory [13] into Eq. (4.33) the dynamic vertex function is

$$\mathring{f}_{t\tilde{t}}^{(d)}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=(\kappa l)^2\lambda_t(l)\times\{1+E_t(v(l),w(l),\{\Xi(l)\})\}, \quad (5.2)$$

where $\{\Xi(l)\}$ characterizes the renormalized set of variables

$$\{\Xi(l)\}=\{\gamma_q(l),u(l),\Gamma(l),L_\phi(l),\lambda(l),\lambda_l(l),\lambda_t(l),g(l),g_l(l)\}.$$

The parameters $v(l)$ and $w(l)$ represent the renormalized temperature and frequency variables defined by

$$v(l)=\frac{\xi^{-2}(t)}{(\kappa l)^2}, \quad w(l)=\frac{\Omega(l)}{(\kappa l)^4}=\frac{\omega}{2\Gamma(l)(\kappa l)^4}. \quad (5.3)$$

The complex function E_t contains the contributions from the loop expansion. Defining

$$f_t(l)=\frac{g(l)}{\sqrt{\Gamma(l)\lambda_t(l)}}, \quad (5.4)$$

quite analogous to its unrenormalized counterpart in Eq. (3.48), the ϵ-expanded one loop contribution to the shear viscosity is

$$E_t(v(l),w(l),\{\Xi(l)\})=-\frac{f_t^2}{96}\Bigg\{1+6\left[i\frac{v^2}{w}\ln v+\frac{1}{v_+-v_-}\left(\frac{v_-^2}{v_+}\ln v_--\frac{v_+^2}{v_-}\ln v_+\right)\right]-\frac{4}{(v_+-v_-)^3}\Bigg[\frac{v_+^3-v_-^3}{3}+\frac{3}{2}(v_+-v_-)$$
$$\times(v_+^2\ln v_++v_-^2\ln v_-)-(v_+^3\ln v_+-v_-^3\ln v_-)\Bigg]+\frac{2}{(v_+-v_-)^2}\Bigg[\frac{v_+^3}{v_-}(1+4\ln v_+)+\frac{v_-^3}{v_+}(1+4\ln v_-)$$
$$+\left(\frac{1}{v_-}-\frac{2}{v_+-v_-}\right)\frac{v_+^4\ln v_+-v^4\ln v}{v_-}+\left(\frac{1}{v_+}+\frac{2}{v_+-v_-}\right)\frac{v_-^4\ln v_--v^4\ln v}{v_+}\Bigg]\Bigg\}, \quad (5.5)$$

where we have dropped the argument l in the parameters on the right hand side of the equation. The parameters introduced in Eq. (5.5) are defined as

$$v_\pm(l)=\frac{v(l)}{2}\pm\sqrt{\left(\frac{v(l)}{2}\right)^2+iw(l)}. \quad (5.6)$$

Inserting Eqs. (5.2) and (2.11) into Eq. (5.1) the shear viscosity generally reads

$$\overline{\eta(t,\omega)}=\frac{1}{RT}(\kappa l)^2\lambda_t(l)[1+E_t(v(l),w(l),\{\Xi(l)\})], \quad (5.7)$$

leading to the real friction coefficient

$$\beta(t,\omega)\approx(\kappa l)^2\lambda_t(l)[1+(\mathrm{Re}+\mathrm{Im})\times[E_t(v(l),w(l),\{\Xi(l)\})]]. \quad (5.8)$$

The flow of the Onsager coefficient λ_t in Eq. (5.7) is determined by the flow equation (4.38) whose solution may be written as

$$\lambda_t(l)=\lambda_t l^{-2}e^{\int_1^l(dx/x)\zeta_{\lambda_t}}. \quad (5.9)$$

The flow parameter l, the reduced temperature t, and the frequency ω are related by a matching condition introduced in the following section, which defines the function $l(t,\omega)$.

B. Sound velocity and sound attenuation

The sound velocity and the sound attenuation are obtained from Eqs. (3.25) and (3.26) by inserting the renormalized parameters into

$$\mathcal{C}_s^2(t,\omega)=-\mathring{g}_{q\tilde{l}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})\mathring{g}_{l\tilde{q}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\}), \quad (5.10)$$

$$\mathcal{D}_s(t,\omega)=\mathring{f}_{q\tilde{q}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})+\mathring{f}_{l\tilde{l}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\}). \quad (5.11)$$

Separating static and dynamic functions with Eq. (3.30) we get

$$\mathcal{C}_s^2(t,\omega)=-a_j\mathring{\Gamma}_{qq}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})[\mathring{g}^{(d)}_{q\tilde{l}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})]^2, \quad (5.12)$$

$$\mathcal{D}_s(t,\omega)=\mathring{\Gamma}_{qq}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})\mathring{f}^{(d)}_{q\tilde{q}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})+a_j\mathring{f}^{(d)}_{l\tilde{l}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\}). \quad (5.13)$$

With Eq. (4.32) the solution of the corresponding renormalization group equation for $g^{(d)}_{q\tilde{l}}$ is

$$\mathring{g}^{(d)}_{q\tilde{l}}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=(\kappa l)^3Z_q^{-1/2}e^{(1/2)\int_1^l(dx/x)\zeta_q}\times\hat{g}^{(d)}_{q\tilde{l}}(v(l),w(l),\{\Xi(l)\}). \quad (5.14)$$

The amplitude function simply reads

$$\hat{g}^{(d)}_{q\tilde{l}}(v(l),w(l),\{\Xi(l)\})=ic(l). \quad (5.15)$$

At finite frequencies no new ϵ poles appear in the vertex functions, thus the renormalization of $\mathring{\Gamma}_{qq}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})$ is the same as for the corresponding static vertex function in Eqs. (4.12) and (4.14). Therefore we have

$$\mathring{\Gamma}_{qq}(\xi^{-2},\mathring{\Omega},\{\mathring{\Xi}\})=Z_q^{-1}e^{\int_1^l(dx/x)\zeta_q}\hat{\Gamma}_{qq}(v(l),w(l),\{\Xi(l)\}). \quad (5.16)$$

The amplitude function may be written as

$$\hat{\Gamma}_{qq}(v(l),w(l),\{\Xi(l)\})=\frac{a_q}{1+[\gamma_q^2(l)/a_q]F_+(v(l),w(l),\{\Xi(l)\})}, \quad (5.17)$$

where the function F_+ contains the contributions of the loop expansion. In one loop order the ϵ-expanded amplitude function is

$$F_+(v(l),w(l),\{\Xi(l)\}) = -\frac{1}{4}\left\{\frac{v^2}{v_+v_-}\ln v + \frac{1}{v_+-v_-}\left[\frac{v_-^2}{v_+}\ln v_- - \frac{v_+^2}{v_-}\ln v_+\right]\right\}. \tag{5.18}$$

The amplitude function F_+ reduces at vanishing frequency to the amplitude function of the static ϕ^2-ϕ^2 correlation function

$$\lim_{\omega\to 0} F_+(v(l),w(l),\{\Xi(l)\}) = F_+^{(s)}\left(\frac{\xi^{-2}(t)}{(\kappa l)^2},u(l)\right). \tag{5.19}$$

This reflects the fact that the isochoric specific heat and the adiabatic compressibility have the same weak singularity. Inserting Eqs. (5.14)–(5.17) into Eq. (5.12) we get

$$\mathcal{C}_s^2(t,\omega) = \frac{a_j a_q(\kappa l)^6 c^2(l)}{1+[\gamma_q^2(l)/a_q]F_+(v(l),w(l),\{\Xi(l)\})}. \tag{5.20}$$

The dynamic vertex functions in Eq. (5.13) are given by

$$\mathring{\Gamma}_{q\tilde{q}}^{(d)}(\mathring{r},\mathring{\Omega},\{\mathring{\Xi}\}) = (\kappa l)^2 Z_q e^{-\int_1^l (dx/x)\zeta_q}\lambda(l), \tag{5.21}$$

$$\mathring{\Gamma}_{l\tilde{l}}^{(d)}(\mathring{r},\mathring{\Omega},\{\mathring{\Xi}\}) = (\kappa l)^2\lambda_l(l)\{1+E_l(v(l),w(l),\{\Xi(l)\})\}. \tag{5.22}$$

With Eqs. (5.21) and (5.22) the complex coefficient (5.13) becomes

$$\mathcal{D}_s(t,\omega) = \frac{a_q(\kappa l)^2\lambda(l)}{1+[\gamma_q^2(l)/a_q]F_+(v(l),w(l),\{\Xi(l)\})} + a_j(\kappa l)^2\lambda_l(l)\{1+E_l(v(l),w(l),\{\Xi(l)\})\}, \tag{5.23}$$

where Eqs. (5.16) and (5.17) also have been used. The function E_l includes the contributions from perturbation expansion and reads in one loop order (ϵ expanded)

$$\begin{aligned} E_l(v(l),w(l),\{\Xi(l)\}) = -\frac{f_l^2}{72}\Bigg\{ & 1+6\left[\frac{iv^2}{w}\ln v - \frac{1}{v_+-v_-}\left(\frac{v_-^2}{v_+}\ln v_- - \frac{v_+^2}{v_-}\ln v_+\right)\right] - \frac{v_+^3-v_-^3}{(v_+-v_-)^3} + \frac{3}{2}\frac{v_+^3\ln v_+ - v_-^3\ln v_-}{(v_+-v_-)^3} \\ & -\frac{9}{2}\frac{v_+^2\ln v_+ - v_-^2\ln v_-}{(v_+-v_-)^2} + \frac{3}{4}\frac{1}{(v_+-v_-)^2}\left[\frac{ivw}{v_+-v_-} + \frac{v_+^3}{v_-}(1+4\ln v_+) + \frac{v_-^3}{v_+}(1+4\ln v_-) - v^2\right] \\ & +\frac{3}{4}\frac{1}{(v_+-v_-)^2}\left[\left(\frac{1}{v_-}-\frac{2}{v_+-v_-}\right)\frac{v_+^4\ln v_+ - v^4\ln v}{v_-} + \left(\frac{1}{v_+}+\frac{2}{v_+-v_-}\right)\frac{v_-^4\ln v_- - v^4\ln v}{v_+}\right] \\ & +\frac{3}{4}\frac{v}{(v_+-v_-)^2}\left[\frac{v_+^2}{v_-}(1+3\ln v_+) + \frac{v_-^2}{v_+}(1+3\ln v_-)\right] + 4\frac{v}{(v_+-v_-)^2}\left[\left(\frac{1}{v_-}-\frac{2}{v_+-v_-}\right)\right. \\ & \left.\times\frac{v_+^3\ln v_+ - v^3\ln v}{v_-} + \left(\frac{1}{v_+}+\frac{2}{v_+-v_-}\right)\frac{v_-^3\ln v_- - v^3\ln v}{v_+}\right]\Bigg\}. \end{aligned} \tag{5.24}$$

The flow of the parameters $\lambda(l)$, $\lambda_l(l)$, and $c(l)$ is determined by Eqs. (4.36), (4.38), and (4.41) with the solutions

$$\lambda(l) = \lambda l^{-2} e^{\int_1^l (dx/x)\zeta_q}, \quad \lambda_l(l) = \lambda_l l^{-2} e^{\int_1^l (dx/x)\zeta_{\lambda_l}},$$
$$c(l) = c l^{-3} e^{(1/2)\int_1^l (dx/x)\zeta_q}. \tag{5.25}$$

The sound velocity and the sound diffusion coefficient are determined by inserting Eqs. (5.20) and (5.23) into Eq. (3.20). The sound attenuation is then defined by

$$\alpha(t,\omega) = \frac{\omega^2}{2c_s^3(t,\omega)} D_s(t,\omega). \tag{5.26}$$

In the expressions obtained so far for the shear viscosity [Eq. (5.7)], the sound velocity [Eq. (5.20)], and the diffusion [Eq. (5.23)] the relation among the flow parameter, the reduced temperature, and the frequency needs to be specified. There are several conditions that restrict the choice of this relation. First the amplitude functions (5.5), (5.18), and (5.24) have to stay finite at the limits $t\to 0$ at fixed ω and also in the limit $\omega\to 0$ at fixed t. At last in the zero frequency limit the relation has to reduce to the condition (4.15) used in statics. At finite frequencies an appropriate choice for the matching condition is

$$\left|\left(\frac{\xi^{-2}(t)}{2(\kappa l)^2}\right)^2 + iw(l)\right| = \frac{1}{4}, \tag{5.27}$$

where $|\cdots|$ denotes the length in the complex plane. This choice is suggested by the appearance of the frequency dependence in all our expressions through the square root in Eq. (5.6). Taking the square of Eq. (5.27) and inserting the definition (5.3) of $w(l)$ the relation becomes

$$\xi^{-8}(t)+\left(\frac{2\omega}{\Gamma(l)}\right)^2=(\xi_0^{-1}l)^8. \tag{5.28}$$

For any fixed frequency Eq. (5.27) implicitly defines a flow parameter $l(t,\omega)$. In contrast to the zero frequency case the critical limit $\xi^{-1}(t)\to 0$ at fixed frequency corresponds to a finite flow parameter value $l_c(\omega)$, which is defined by the equation

$$(\xi_0^{-1}l_c)^8=\left(\frac{2\omega}{\Gamma(l_c)}\right)^2. \tag{5.29}$$

VI. CRITICAL BEHAVIOR AT ZERO FREQUENCY

A. Shear viscosity and thermal conductivity

In order to obtain the initial values of the dynamic model parameters $\Gamma(l_0)$ and $f_t(l_0)$, appearing in the sound velocity, the attenuation, and the friction coefficient, experiments on the thermal conductivity and/or shear viscosity performed at $\omega=0$ have to be fitted with our theoretical expressions. From the definition of the thermal diffusion coefficient $D_T=\kappa_T/(\rho C_P)$ and Eq. (3.27) the thermal conductivity is given by

$$\kappa_T(t)=\rho C_P(t)\mathring{f}_{\phi\tilde{\phi}}(\xi^{-2},\{\mathring{\Xi}\}). \tag{6.1}$$

Because $\mathring{f}_{\phi\tilde{\phi}}$ in Eq. (6.1) is a real function at zero frequency we have omitted the real part. With Eq. (3.30) we may write

$$\kappa_T(t)=\rho C_P(t)\mathring{\hat{\Gamma}}^{(s)}_{\phi\phi}(\xi^{-2},\mathring{u})\mathring{f}^{(d)}_{\phi\tilde{\phi}}(\xi^{-2},\{\mathring{\Xi}\}). \tag{6.2}$$

The solution of the renormalization group equation for the dynamic order parameter vertex function inserted into Eq. (4.30) reads

$$\mathring{f}^{(d)}_{\phi\tilde{\phi}}(\xi^{-2},\{\mathring{\Xi}\})=Z_\phi e^{-\int_1^l(dx/x)\zeta_\phi}\Gamma(l)\{1+G(\{\Xi(l)\})\}, \tag{6.3}$$

where the function G contains the contributions from the perturbation expansion. The flow of the Onsager coefficient Γ is determined by the corresponding flow equation (4.36) from which the solution

$$\Gamma(l)=\Gamma e^{\int_1^l(dx/x)(\zeta_\phi+\zeta_\Gamma^{(d)})} \tag{6.4}$$

follows. Together with the static vertex function (4.17) the thermal conductivity is

$$\kappa_T(t)=\rho C_P(t)\xi^{-2}(t)\Gamma(t)\hat{\Gamma}_{\phi\phi}(u(t))[1+G(\{\Xi(t)\})]. \tag{6.5}$$

The flow parameter l has been replaced by the reduced temperature at zero frequency using the relation (4.16). The shear viscosity (5.7) reduces at vanishing frequency to

$$\overline{\eta}(t)=\frac{1}{RT}\xi^{-2}(t)\lambda_t(t)[1+E_t(\{\Xi(t)\})]. \tag{6.6}$$

At vanishing frequency the amplitude functions ϵ expanded in one loop order are

$$G(\{\Xi\})=-\frac{f_t^2}{16},\quad E_t(\{\Xi\})=-\frac{f_t^2}{36}. \tag{6.7}$$

Equations (6.5)–(6.7) will be used in the second part of this work to analyze experimental results of the thermal conductivity and the shear viscosity in several liquids [27]. In the critical limit the specific heat and the correlation length behave as

$$C_P(t)=A_+t^{-\gamma},\quad \xi(t)=\xi_0t^{-\nu}, \tag{6.8}$$

where γ and ν are the critical exponents. The relation between flow parameter and reduced temperature (4.16) is given by $l=t^\nu$. In the asymptotic limit the flow of the Onsager coefficients Γ and λ_t in Eqs. (6.4) and (5.9) reduces to simple power laws:

$$\Gamma(l)=\Gamma l^{\zeta_\phi^\star+\zeta_\Gamma^{(d)\star}},\quad \lambda_t(l)=\lambda_t l^{-2+\zeta_{\lambda_t}^\star}. \tag{6.9}$$

Introducing the notation of Siggia, Halperin, and Hohenberg [1], that is, $x_\lambda\equiv-\zeta_\Gamma^{(d)\star}$ and $x_\eta\equiv-\zeta_{\lambda_t}^\star$ and using Eq. (4.10), the asymptotic behavior of the Onsager coefficients is

$$\Gamma(t)=\Gamma t^{-\nu(\eta+x_\lambda)},\quad \lambda_t(t)=\lambda_t t^{-\nu(2+x_\eta)}. \tag{6.10}$$

Inserting Eqs. (6.8)–(6.10) into Eqs. (6.5) and (6.6) we obtain the asymptotic behavior of the thermal conductivity and shear viscosity,

$$\kappa_T^{(as)}(t)=\kappa_T^{(c)}t^{-\nu x_\lambda},\quad \overline{\eta^{(as)}}(t)=\overline{\eta^{(c)}}t^{-\nu x_\eta} \tag{6.11}$$

with the amplitudes

$$\kappa_T^{(c)}=\rho A_+\xi_0^{-2}\Gamma\hat{\Gamma}^{(s)}_{\phi\phi}(u^\star)[1+G(\{\Xi^\star\})], \tag{6.12}$$

$$\overline{\eta^{(c)}}=\frac{1}{RT}\xi_0^{-2}\lambda_t[1+E_t(\{\Xi^\star\})]. \tag{6.13}$$

In the asymptotic form (6.11) of the thermal conductivity the critical exponent relation $\gamma=\nu(2-\eta)$ has been used. The dynamic ζ functions are in one loop order

$$\zeta_\Gamma^{(d)}=-\frac{3}{4}f_t^2,\quad \zeta_{\lambda_t}=-\frac{1}{24}f_t^2. \tag{6.14}$$

From Eq. (4.40) it follows that at the fixed point $\zeta_\phi^\star+\zeta_\Gamma^{(d)\star}+\zeta_{\lambda_t}^\star=-\epsilon$. With Eq. (6.14) we get the one loop fixed point $f_t^\star=\sqrt{24\epsilon/19}$. Inserting this value into Eq. (6.14) the one loop values of the dynamic exponents are

$$x_\lambda=\frac{18}{19}\epsilon,\quad x_\eta=\frac{\epsilon}{19}. \tag{6.15}$$

At $d=3$ ($\epsilon=1$) the values are $x_\lambda=0.947$ and $x_\eta=0.053$ where the latter is in good agreement with results obtained from the mode coupling theory [28] and the decoupled mode theory for binary liquids at the consolute point [29] and for pure liquids [30].

B. Sound attenuation and sound velocity

From Eqs. (5.20) and (5.23) one immediately sees together with Eq. (5.19) that the complex functions $\mathcal{C}_s$ and $\mathcal{D}_s$ in the limit $\omega \to 0$ reduce to the real coefficients

$$\mathcal{C}_s^2(t) = \frac{a_j a_q \mathring{c}^2 Z_q^{-1} e^{\int_1^l (dx/x)\zeta_q}}{1+[\gamma_q^2(l)/a_q]F_+^{(s)}(u(l))}, \tag{6.16}$$

$$\mathcal{D}_s(t) = \frac{a_q \mathring{\lambda} Z_q^{-1} e^{\int_1^l (dx/x)\zeta_q}}{1+[\gamma_q^2(l)/a_q]F_+^{(s)}(u(l))} + a_j(\kappa l)^2 \lambda_l(l) \times \{1+E_l(\{\Xi(l)\})\}, \tag{6.17}$$

where we have used Eqs. (4.24), (4.29), (4.34), and (5.25) for the parameters $\lambda(l)$, and $c(l)$. With Eqs. (2.14), (2.17) and (4.18) from statics we may write

$$Z_q^{-1} e^{\int_1^l (dx/x)\zeta_q} = \frac{1}{a_q RT\rho}\left(\frac{\partial P}{\partial \rho}\right)_\sigma \left(1+\frac{\gamma_q^2(l)}{a_q}F_+^{(s)}(u(l))\right). \tag{6.18}$$

Eliminating the renormalization factors with (6.18) and inserting the definitions of $\mathring{c}$ and a_j from Eqs. (2.7) and (2.11) we get

$$\mathcal{C}_s^2(t) = \left(\frac{\partial P}{\partial \rho}\right)_\sigma, \tag{6.19}$$

$$\mathcal{D}_s(t) = \frac{\mathring{\lambda}}{RT\rho}\left(\frac{\partial P}{\partial \rho}\right)_\sigma + (\kappa l)^2 \lambda_l(l)[1+E_l(\{\Xi(l)\})]. \tag{6.20}$$

From Eq. (6.19) one can see that $\mathcal{C}_s$ is expressed by the same thermodynamic derivative as the hydrodynamic sound velocity (3.8). The difference is that the thermodynamic derivative in Eqs. (6.19) and (6.20) now contains the critical singularity. In the asymptotic region we may write

$$\left(\frac{\partial \rho}{\partial P}\right)_\sigma = B_+ t^{-\alpha} \tag{6.21}$$

and therefore the sound velocity $c_s^{(as)} = \mathcal{C}_s = B_+^{-1/2} t^{\alpha/2}$ vanishes with the critical exponent $\alpha/2$. The one loop amplitude function (5.24) reduces at $\omega = 0$ to

$$E_l(\{\Xi\}) = -\frac{f_l^2}{48}. \tag{6.22}$$

With Eqs. (5.25) and (6.21) the critical behavior of Eq. (6.20) is

$$\mathcal{D}_s(t) = \frac{\mathring{\lambda} B_+}{RT\rho} t^{\alpha} + \lambda_l[1+E_l(\{\Xi^\star\})] t^{-\nu x_l}. \tag{6.23}$$

The first term vanishes for $T \to T_c$, the second term diverges like $t^{-\nu x_l}$ where x_l is defined as $x_l \equiv -\zeta_{\lambda_l}^\star$ analogous to the transverse mode. In one loop order the ζ function reads

$$\zeta_{\lambda_l} = -\frac{f_l^2}{12}. \tag{6.24}$$

The momentum density ζ functions fulfill the fixed point relation (4.43) from which it follows that the dynamic exponent of the second part in Eq. (6.23) $x_l = x_\eta$ is the same as for the shear viscosity. Thus Eq. (6.23) may be given as an asymptotic power law

$$\mathcal{D}_s^{(as)}(t) = D_2^{(c)} t^{-\nu x_\eta}, \quad D_2^{(c)} = \lambda_l[1+E_l(\{\Xi^\star\})]. \tag{6.25}$$

Inserting Eq. (5.25) and the unrenormalized parameters (2.8) and (2.9) into Eq. (6.20) and using the thermodynamic relation

$$\left(\frac{\partial \rho}{\partial \sigma}\right)_P^2 \left(\frac{\partial P}{\partial \rho}\right)_\sigma = \rho^2 T\left(\frac{1}{C_V} - \frac{1}{C_P}\right),$$

the coefficient $\mathcal{D}_s$ at vanishing frequency can be written as

$$\mathcal{D}_s(t) = \frac{\kappa_T^{(0)}}{\rho}\left(\frac{1}{C_V(t)} - \frac{1}{C_P(t)}\right) + \frac{\zeta^{(0)} + \frac{4}{3}\overline{\eta^{(0)}}}{\rho} Z_{\lambda_l}^{-1} e^{\int_1^{l(t)} (dx/x)\zeta_{\lambda_l}}[1+E_l(\{\Xi(t)\})]. \tag{6.26}$$

The above sound diffusion coefficient looks like the hydrodynamic expression in Eq. (3.8) separating into a part caused by the finite thermal conductivity and a part caused by the finite shear and bulk viscosity. In the asymptotic limit contributions due to critical fluctuations appear that are described by the two critical specific heats in the first term of Eq. (6.26) and the exponential factor in the second term. We want to emphasize that in Eq. (6.26) the thermal conductivity, the shear viscosity, and the bulk viscosity only enter by their background values. It would be erroneous to take the hydrodynamic expression in Eq. (3.8) and simply insert the asymptotic power laws for all appearing transport coefficients and thermodynamic derivatives.

To get the sound diffusion coefficient D_s at vanishing frequency we have to insert Eqs. (5.20) and (5.23) into Eq. (3.20) and take the limit $\omega \to 0$. The result is

$$D_s(t) = -\lim_{\omega \to 0} \frac{\mathrm{Im}[\mathcal{C}_s^2 - i\omega \mathcal{D}_s]}{\omega} = \frac{\gamma_q^2(l)}{2a_q \Gamma(l)(\kappa l)^4}\left(\frac{\partial P}{\partial \rho}\right)_\sigma \frac{F'_+(\{\Xi(l)\})}{1+[\gamma_q^2(l)/a_q]F_+^{(s)}(u(l))} + \mathcal{D}_s(t) \tag{6.27}$$

with the flow parameter $l = l(t)$ from Eq. (4.16) and

$$F'_{+}(\{\Xi(l)\})=\lim_{\omega\to 0}\frac{\partial F_{+}(v(l),w(l),\{\Xi(l)\})}{\partial w(l)}. \quad (6.28)$$

Inserting the matching condition (4.16) and the asymptotic power laws (6.10) for the Onsager coefficient and (6.25) for $\mathcal{D}_s(t)$, the sound diffusion coefficient can be written as

$$D_s(t)=D_1^{(c)}t^{-z\nu+\alpha}+D_2^{(c)}t^{-\nu x_\eta}, \quad (6.29)$$

where we have introduced the exponent

$$z=4-\eta-x_\lambda \quad (6.30)$$

and the amplitude

$$D_1^{(c)}=\frac{\gamma_q^{\star 2}\xi_0^4}{2a_q\Gamma B_+}\frac{\mathrm{Re}[F'_{+}(\{\Xi^\star\})]}{1+[\gamma_q^{\star 2}/a_q]F_{+}^{(s)}(u^\star)}. \quad (6.31)$$

In Eq. (6.30) the exponent of the critical thermal conductivity x_λ enters. With the relation $x_\lambda+x_\eta=\epsilon-\eta$ [1] between the dynamic exponents, z may be expressed by the exponent of the critical shear viscosity

$$z=4-\epsilon+x_\eta. \quad (6.32)$$

The first term in Eq. (6.29) contains the leading strong singularity with $z=3.053$ in one loop order. The second term diverges only weakly with the exponent of the shear viscosity. From Eqs. (5.26) and (6.21) one can see that the asymptotic behavior of the sound attenuation at small frequencies is

$$\alpha^{(as)}(t,\omega)=\alpha^{(c)}\omega^2 t^{-(z\nu+\alpha/2)}, \quad (6.33)$$

which is in agreement with results obtained in [2,3].

VII. CRITICAL BEHAVIOR AT T_c AND SCALING FUNCTION

A. Viscosity and sound at T_c

At the critical temperature T_c the flow parameter is uniquely related to the frequency by Eq. (5.29). At small frequencies in the asymptotic region the Onsager coefficient Γ behaves as $\Gamma(l_c)=\Gamma l_c^{-(x_\lambda+\eta)}=\Gamma l_c^{z-4}$ according to Eq. (6.9). We used the definition of the dynamical exponent z by Eq. (6.30). Replacing l by its solution of the matching condition Eq. (5.29)

$$l_c=(b\omega)^{1/z}, \quad b=\frac{2\xi_0^4}{\Gamma}, \quad (7.1)$$

the complex shear viscosity (5.7) for frequencies in the asymptotic region of reads

$$\overline{\eta(}0,\omega)=\frac{\lambda_t}{\xi_0^2 RT}[1+E_t(\{\Xi^\star\})]l_c^{-x_\eta}. \quad (7.2)$$

Note that Eq. (5.29) may be rewritten as $w(l_c)=\frac{1}{4}$ by inserting the definition (5.3). The amplitude function $E_t(\{\Xi^\star\})$ at $t=0$ is the complex function

$$E_t(\{\Xi^\star\})=\frac{f_t^{\star 2}}{288}\left[1+6\ln 2+i\frac{3\pi}{2}\right]. \quad (7.3)$$

With Eq. (7.1) the frequency dependence of the shear viscosity at the critical point is

$$\overline{\eta(}0,\omega)=\overline{\eta^{(\omega)}}\omega^{-x_\eta/z}. \quad (7.4)$$

This result for the asymptotic shear viscosity is in agreement with the result of the decoupled mode theory of Bhattacharjee and Ferrell [30].

The complex functions Eqs. (5.20) and (5.23) appearing in the sound mode turn in the asymptotic frequency region to

$$\mathcal{C}_s^2(0,\omega)=\frac{a_j a_q c^2\xi_0^{-6}}{1+(\gamma_q^{2\star}/a_q)F_{+}(\{\Xi^\star\})}l_c^{\alpha/\nu}, \quad (7.5)$$

$$\mathcal{D}_s(0,\omega)=\frac{a_q\xi_0^{-2}\lambda}{1+(\gamma_q^{2\star}/a_q)F_{+}(\{\Xi^\star\})}l_c^{\alpha/\nu} + a_j\xi_0^{-2}\lambda_l[1+E_l(\{\Xi^\star\})]l_c^{-x_\eta}, \quad (7.6)$$

where the asymptotic power laws for the static and dynamic parameters $c(l_c)=cl_c^{-3+\alpha/\nu}$, $\lambda(l_c)=\lambda l_c^{-2+\alpha/\nu}$, and $\lambda_l(l_c)=\lambda_l l_c^{-(2+x_\eta)}$ follow from Eqs. (4.10) and (5.25). The amplitude function $F_{+}(\{\Xi^\star\})$ at $t=0$ reads

$$F_{+}(\{\Xi^\star\})=\frac{1}{4}\left[\ln 2+i\frac{\pi}{4}\right]. \quad (7.7)$$

From Eqs. (3.20) and (7.1) we get for the sound velocity

$$c_s^2(0,\omega)=c_1^2\omega^{\alpha/\nu z}+c_2^2\omega^{1+\alpha/\nu z}+c_3^2\omega^{1-x_\eta/z}. \quad (7.8)$$

At small frequencies the first term in Eq. (7.8) is the leading one due to the small positive exponent. Further, the amplitudes c_2 and c_3 are small compared to the amplitude of the first term. Thus the sound velocity at the critical point has the form

$$c_s(0,\omega)\sim c_1\omega^{\alpha/2\nu z}. \quad (7.9)$$

Analogous to Eq. (7.8) for the asymptotic behavior at small frequencies of the sound diffusion follows

$$D_s(0,\omega)=D_1\omega^{-1+\alpha/\nu z}+D_2\omega^{\alpha/\nu z}+D_3\omega^{-x_\eta/z}. \quad (7.10)$$

In the limit $\omega\to 0$ the first term in Eq. (7.6) shows a strong divergence while the last term has a weak divergence. For asymptotic small frequencies the sound diffusion coefficient is described by the single power law

$$\mathcal{D}_s(0,\omega)\sim D_1\omega^{-1+\alpha/\nu z}. \quad (7.11)$$

Inserting the power laws (7.9) and (7.11) into the definition of the sound attenuation (5.26) we get finally

$$\alpha(0,\omega)=\alpha^{(\omega)}\omega^{1-\alpha/2\nu z} \quad (7.12)$$

in agreement with [5].

B. Scaling functions for the viscosity and sound

The connection between flow parameter and reduced temperature (5.28) offers the possibility to define a scaling variable. Inserting the asymptotic relations $\xi(t)=\xi_0 t^{-\nu}$, $\Gamma(l)=\Gamma l^{z-4}$ and defining the variable

$$y=\frac{\omega}{2\Gamma\xi_0^{-4}t^{\nu z}}. \tag{7.13}$$

The matching condition Eq. (5.28) may be written after some rearrangement

$$(t^{-\nu}l)^8=1+16y^2(t^{-\nu}l)^{2(4-z)}. \tag{7.14}$$

This form of the equation holds for all liquids because all parameters that characterize a special liquid are absorbed in y. Equation (7.14) also shows that the product $t^{-\nu}l$ is a function $S(y)=t^{-\nu}l$ of y alone, where S is determined implicitly by solving Eq. (7.14). The flow parameter may be written as

$$l=t^{\nu}S(y). \tag{7.15}$$

At zero frequency ($\omega=0$) the solution of the matching condition (7.14) is equal to the asymptotic form of Eq. (4.15), thus $S(0)=1$. For large y one may neglect the 1 on the right hand side of Eq. (7.14) and we obtain $S(y)=(4y)^{1/z}$ reproducing the power laws in frequency at T_c. Inserting Eqs. (7.13) and (7.15) into the definition (5.3), the parameters v and w may be expressed entirely by y via

$$v(l)=[S(y)]^{-2}, \quad w(l)=y[S(y)]^{-z}. \tag{7.16}$$

Inserting this into the amplitude functions F_+, E_l, and E_t they become complex functions of y only. We introduce the function

$$F_+(v(l),w(l),\{\Xi^\star\})=F_+([S(y)]^{-2},y[S(y)]^{-z},\{\Xi^\star\}) \equiv \hat{F}_+(y) \tag{7.17}$$

and $\hat{E}_t(y)$ and $\hat{E}_l(y)$ accordingly. We then define the complex scaling function for the shear viscosity by extracting the asymptotic form (6.11)

$$\overline{\eta(t,\omega)}=\overline{\eta^{(as)}}(t)Y_\eta(y), \tag{7.18}$$

where the scaling function $Y_\eta(y)$ is normalized to 1 at $y=0$. From Eq. (5.7) we have

$$Y_\eta(y)=S(y)^{-x_\eta}\frac{1+\hat{E}_t(y)}{1+E_t(\{\Xi^\star\})}. \tag{7.19}$$

This expression might be compared with the result of [9]. Following the manipulations in that paper, which make use of the smallness of x_η, we identify the impedance function (sum of the real and imaginary parts of the shear viscosity) as

$$H=\ln(B_\infty y)-z\ln[S(y)]+\frac{z}{x_\eta}\{\mathrm{Re}[\hat{E}_t(y)]-\mathrm{Re}[E_t(\{\Xi^\star\})] + \mathrm{Im}[\hat{E}_t(y)]\}. \tag{7.20}$$

The first term comes from the division by the large y behavior, which cancels the ln(y) terms of the following terms. A more detailed comparison will be given in the third part of this work.

Let us turn to the sound velocity and attenuation. The complex coefficient (5.20) and (5.23) can be written with Eq. (7.16) as

$$\mathcal{C}_s^2(t,\omega)=a_j a_q c^2\xi_0^{-6}\frac{t^{\alpha}[S(y)]^{\alpha/\nu}}{1+(\gamma_q^{2\star}/a_q)\hat{F}_+(y)}, \tag{7.21}$$

$$\mathcal{D}_s(t,\omega)=a_q\xi_0^{-2}\lambda\frac{t^{\alpha}S^{\alpha/\nu}(y)}{1+(\gamma_q^{2\star}/a_q)\hat{F}_+(y)}+a_j\xi_0^{-2}\lambda_l \times[1+\hat{E}_l(y)]t^{-\nu x_\eta}[S(y)]^{-x_\eta}. \tag{7.22}$$

From the real and imaginary parts of Eqs. (7.21) and (7.22) one may calculate the squared sound velocity and the sound diffusion coefficient defined in Eq. (3.20). We note that the imaginary part of F_+ is proportional to the frequency. Thus we have

$$\mathrm{Im}[F_+(v(l),w(l),\{\Xi^\star\})]=w(l)\mathrm{Im}[\overline{F_+}(v(l),w(l),\{\Xi^\star\})]=y[S(y)]^{-z}\mathrm{Im}[\hat{\overline{F}}_+(y)] \tag{7.23}$$

with a finite function $\mathrm{Im}[\hat{\overline{F}}_+(y)]$ at $y=0$. In the vicinity of the fixed point the sound velocity and diffusion coefficient read

$$c_s^2(t,\omega)=a_j a_q c^2\xi_0^{-6}\frac{1+(\gamma_q^{2\star}/a_q)\mathrm{Re}[\hat{F}_+(y)]}{|1+(\gamma_q^{2\star}/a_q)\hat{F}_+(y)|^2}t^{\alpha}[S(y)]^{\alpha/\nu}+2a_q\Gamma\lambda\xi_0^{-6}\frac{(\gamma_q^{2\star}/a_q)\mathrm{Im}[\hat{\overline{F}}_+(y)]}{|1+(\gamma_q^{2\star}/a_q)\hat{F}_+(y)|^2}t^{\alpha+\nu z}y^2[S(y)]^{-z+\alpha/\nu} + 2a_j\Gamma\lambda_l\xi_0^{-6}\mathrm{Im}[\hat{E}_l(y)]t^{\nu(z-x_\eta)}y[S(y)]^{-x_\eta}, \tag{7.24}$$

$$D_s(t,\omega)=-\frac{a_j a_q c^2\xi_0^{-2}(\gamma_q^{2\star}/a_q)\mathrm{Im}[\hat{\overline{F}}_+(y)]}{2\Gamma|1+\gamma_q^{2\star}/a_q)\hat{F}_+(y)|^2}t^{-\nu z+\alpha}[S(y)]^{-z+\alpha/\nu}+a_q\lambda\xi_0^{-2}\frac{1+(\gamma_q^{2\star}/a_q)\mathrm{Re}[\hat{F}_+(y)]}{|1+(\gamma_q^{2\star}/a_q)\hat{F}_+(y)|^2}t^{\alpha}[S(y)]^{\alpha/\nu} + a_j\lambda_l\xi_0^{-2}\{1+\mathrm{Re}[\hat{E}_l(y)]\}t^{-\nu x_\eta}[S(y)]^{-x_\eta}. \tag{7.25}$$

Taking only the leading terms of the asymptotic behavior and extracting the asymptotic behavior at zero frequency the scaling function for the sound velocity is defined by

$$c_s(t,\omega)=c_s^{(as)}(t)Y_c(y) \tag{7.26}$$

with

$$Y_c(y)=S(y)^{\alpha/2\nu}\left(\frac{1+(\gamma_q^{2^\star}/a_q)\mathrm{Re}[\hat{F}_+(y)]}{1+(\gamma_q^{2^\star}/a_q)F_+^{(s)}(u^\star)}\right)^{1/2}\frac{1+(\gamma_q^{2^\star}/a_q)F_+^{(s)}(u^\star)}{|1+(\gamma_q^{2^\star}/a_q)\hat{F}_+(y)|}. \tag{7.27}$$

The scaling function for the sound attenuation reads accordingly

$$\alpha(t,\omega)=\alpha^{(as)}(t)Y_\alpha(y) \tag{7.28}$$

with

$$Y_\alpha(y)=S(y)^{-z-\alpha/2\nu}\frac{\mathrm{Im}[\widehat{F_+}(y)]}{F'_+(\{\Xi^\star\})}\left(\frac{1+(\gamma_q^{2^\star}/a_q)F_+^{(s)}(u^\star)}{1+(\gamma_q^{2^\star}/a_q)\mathrm{Re}[\hat{F}_+(y)]}\right)^{3/2}\frac{|1+(\gamma_q^{2^\star}/a_q)\hat{F}_+(y)|}{1+(\gamma_q^{2^\star}/a_q)F_+^{(s)}(u^\star)}, \tag{7.29}$$

where we have taken out the asymptotic power law (6.33) at small frequencies.

VIII. OUTLOOK

We have calculated within a nonasymptotic field-theoretical renormalization group theory the frequency-dependent viscosity sound velocity and sound attenuation near the gas-liquid critical point in pure fluids in one loop order. These measurable quantities are determined completely by model H dynamical parameters and thermodynamic derivatives. Taking the initial values for the corresponding flow equations for the order parameter Onsager coefficient $\Gamma(l_0)$, the mode coupling $f_t(l_0)$, and the static coupling $\gamma_q(l)$ from experiment a parameter-free prediction of the complete frequency and temperature dependence of the sound mode and the shear viscosity is possible. The non-asymptotic behavior of the dynamical parameters has already been demonstrated [4,31] and satisfactory agreement with experiment has been reached. In the asymptotic limit one introduces a scaled frequency (depending on $\Gamma(l_0)$ and defines scaling functions for the asymptotic crossover in the frequency ω and temperature t from the t axis to the ω axis in the ω-t plane. This allows one to compare with earlier asymptotic calculations [5,8,9]. These topics are described in more detail in Ref. [27].

APPENDIX A: STATIC FUNCTIONAL

Recently a detailed derivation of a functional describing static critical phenomena in liquid mixtures has been given [10]. Therefore we will sketch only briefly the derivation of a static functional for pure liquids. The starting point is a local equilibrium distribution function:

$$w_{\mathrm{loc}}=\frac{1}{\mathcal{N}}e^{-\int_V d^dx\,\Omega(x)/k_BT(x)} \tag{A1}$$

with temperature $T(x)$, chemical potential $\mu(x)$, and velocity $\boldsymbol{v}(x)$ as external fields. $\Omega(x)$ is the corresponding local thermodynamic potential

$$\Omega(x)=e(x)+e_k(x)-T(x)s(x)-\mu(x)\rho(x)-\boldsymbol{v}(x)\boldsymbol{j}'(x) \tag{A2}$$

in which $e(x)$ is the internal energy density and $e_k(x)=\boldsymbol{j}'(x)^2/2\rho(x)$ is the kinetic energy density. Assuming that the densities are fluctuating about their thermodynamic average values, we can write

$$e(x)=e+\Delta e(x),\quad e_k(x)=e_k+\Delta e_k(x),$$
$$s(x)=s+\Delta s(x),\quad \rho(x)=\rho+\Delta\rho(x), \tag{A3}$$
$$\boldsymbol{j}'(x)=\boldsymbol{j}'+\Delta\boldsymbol{j}'(x).$$

Additionally we allow small variations of the conjugated external fields:

$$T(x)=T+\delta T(x),\quad \mu(x)=\mu+\delta\mu(x),$$
$$\boldsymbol{v}(x)=\boldsymbol{v}+\delta\boldsymbol{v}(x). \tag{A4}$$

Inserting Eqs. (A3) and (A4) into Eq. (A2) the local thermodynamic potential can be split into three parts

$$\frac{\Omega(x)}{k_BT}=\frac{\Omega^{(0)}}{k_BT}+\mathcal{H}(x)-\delta\mathcal{H}(x). \tag{A5}$$

The first part represents the thermodynamic average and contains the Gibb's free energy $\Omega^{(0)}=e+e_k-Ts-\mu\rho-\boldsymbol{v}\cdot\boldsymbol{j}'$. The second part involves the fluctuation contributions and is given by

$$\mathcal{H}(x)=\frac{1}{k_BT}[\Delta e(x)+\Delta e_k(x)-T\Delta s(x)-\mu\Delta\rho(x)$$
$$-\boldsymbol{v}\cdot\Delta\boldsymbol{j}'(x)]. \tag{A6}$$

The third part is the first order contributions due to the external field variation:

$$\delta\mathcal{H}(x)=\frac{e(x)+e_k(x)-\mu\rho(x)-\boldsymbol{v}\cdot\boldsymbol{j}'(x)}{k_BT}\,\frac{\delta T(x)}{T}+\frac{\rho(x)\,\delta\mu(x)}{k_BT}+\frac{\boldsymbol{j}'(x)\cdot\boldsymbol{\delta v}(x)}{k_BT}. \tag{A7}$$

Inserting Eq. (A5) into the local distribution function (A1) and expanding in first order of the external field variations, we get, analogous to [32,33] for the correlation functions,

$$\langle ss\rangle_c=k_BT\left(\frac{\partial s}{\partial T}\right)_\mu,\quad \langle\rho\rho\rangle_c=k_BT\left(\frac{\partial\rho}{\partial\mu}\right)_T, \tag{A8}$$

$$\langle s\rho\rangle_c=k_BT\left(\frac{\partial\rho}{\partial T}\right)_\mu=k_BT\left(\frac{\partial s}{\partial\mu}\right)_T. \tag{A9}$$

From Eqs. (A8) and (A9) one can see that the thermodynamic derivatives involve the chemical potential μ. Experimentally the pressure P is accessible and therefore for a comparison with experimentally measured quantities the local thermodynamic potential (A5) has to be expressed in densities, that correspond to external fields T and P instead of T and μ. This can be obtained by changing from entropy density per volume $s(x)$ to entropy density per mass $\sigma(x)=s(x)/\rho(x)$. The corresponding fluctuations transform like

$$\Delta s(x)=\rho\Delta\sigma(x)+\sigma\Delta\rho(x). \tag{A10}$$

The correlation functions (A8) and (A9) change to

$$\langle\sigma\sigma\rangle_c=\frac{k_BT}{\rho}\left(\frac{\partial\sigma}{\partial T}\right)_P,\quad \langle\rho\rho\rangle_c=\rho k_BT\left(\frac{\partial\rho}{\partial P}\right)_T, \tag{A11}$$

$$\langle\sigma\rho\rangle_c=\frac{k_BT}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P=\rho k_BT\left(\frac{\partial\sigma}{\partial P}\right)_T. \tag{A12}$$

Expanding the Hamiltonian (A6) in powers of the fluctuations of the entropy per mass, the mass density and the momentum density we get

$$H=\int d^dx\left\{\frac{1}{2}a_{\sigma\sigma}[\Delta\sigma(x)]^2+\frac{1}{2}c_{\sigma\sigma}[\boldsymbol{\nabla}\Delta\sigma(x)]^2+\frac{1}{2}a_{\rho\rho}[\Delta\rho(x)]^2+a_{\sigma\rho}\Delta\sigma(x)\Delta\rho(x)+\frac{1}{2}a_j'[\Delta\boldsymbol{j}'(x)]^2+\frac{1}{3!}v_\sigma[\Delta\sigma(x)]^3+\frac{1}{4!}u_\sigma[\Delta\sigma(x)]^4+\frac{1}{2}\gamma_\rho\Delta\rho(x)\times[\Delta\sigma(x)]^2\right\}. \tag{A13}$$

For dynamic calculations it is convenient to choose the entropy density fluctuations as the order parameter. With regard to this we have expanded in Eq. (A13) the entropy density fluctuations up to fourth order, while the mass density and momentum density fluctuations, considered as secondary densities, only need to be expanded up to quadratic order, taking into account in the Hamiltonian all terms relevant for the critical theory. The Gaussian part of Eq. (A13) is nondiagonal and contains terms that are not invariant against order parameter inversion. These terms proportional to $\Delta\rho\Delta\sigma$ and $(\Delta\sigma)^3$ can be removed by introducing a shifted order parameter $\phi_0(x)$ and a transformed secondary density $q_0(x)$ such as

$$\phi_0(x)=\sqrt{N_A}[\Delta\sigma(x)-\langle\Delta\sigma(x)\rangle], \tag{A14}$$

$$q_0(x)=\sqrt{N_A}\left[\Delta\rho(x)-\left(\frac{\partial\rho}{\partial\sigma}\right)_P[\Delta\sigma(x)-\langle\Delta\sigma(x)\rangle]\right]. \tag{A15}$$

N_A has been introduced for convenience to obtain the appearance of the gas constant R instead of the Boltzmann constant k_B in the equations and parameter definitions. Introducing a rescaled momentum density $\boldsymbol{j}=\boldsymbol{j}_l+\boldsymbol{j}_t=\sqrt{N_A}\Delta\boldsymbol{j}'$, one ends up with Eq. (2.10).

APPENDIX B: DYNAMIC EQUATIONS

Due to the critical slowing down the dynamics of critical phenomena is explicitly influenced mainly by slow processes. The influence of variables that vary on short time scales may be considered stochastically. Thus only projections of the dynamic variables into a subspace of slowly varying variables need to be considered [34,35]. Let $\psi_i(x,t)$ be a set of slow variables, then the corresponding dynamic equations can be written as [36–38]

$$\frac{\delta\psi_i(x,t)}{\delta t}=V_i\{\psi(x,t)\}-\sum_j\Lambda_{ij}(x)\frac{\delta H\{\psi(x,t)\}}{\delta\psi_j(x,t)}+\Theta_i(x,t). \tag{B1}$$

$\Theta_i(x,t)$ are fluctuating forces that fulfill the Einstein relations

$$\langle\Theta_i(x,t)\Theta_j(x',t')\rangle=2\Lambda_{ij}(x)\delta(t-t')\delta(x-x') \tag{B2}$$

when Markovian processes are assumed. $\Lambda_{ij}(x)$ are the kinetic coefficients, which are constants $\Lambda_{ij}(x)=\Lambda_{ij}$ in the case of nonconserved densities $\psi(x,t)$ and are given by $\Lambda_{ij}(x)=-\Lambda_{ij}\nabla^2$ in the case of conserved densities. The reversible contributions $V_i\{\psi(x,t)\}$ of the dynamic equations can be written as

$$V_i\{\psi(x,t)\}=\sum_j\int dx'dt'\left[\frac{\delta Q_{ij}(x,t;x',t')}{\delta\psi_j(x',t')}-Q_{ij}(x,t;x',t')\frac{\delta H\{\psi(x,t)\}}{\delta\psi_j(x',t')}\right]. \tag{B3}$$

The quantities $Q_{ij}(x,t;x',t')$ are related to the Poisson brackets of the densities,

$$Q_{ij}(x,t;x',t')=k_BT\{\psi_i(x,t),\psi_j(x',t')\}. \tag{B4}$$

For simple liquids the slowly varying densities $\psi_i(x)$ correspond to the volume densities $s(x)$, $\rho(x)$, and $\boldsymbol{j}'(x)$. Generalized Poisson brackets for hydrodynamic densities may be derived from infinitesimal displacements [39]. The resulting Poisson brackets are

$$\{\boldsymbol{j}'(x,t),s(x',t')\}=s(x,t)\boldsymbol{\nabla}\delta(x-x')\delta(t-t'),$$

$$\{\boldsymbol{j}'(x,t),\rho(x',t')\}=\rho(x,t)\boldsymbol{\nabla}\delta(x-x')\delta(t-t'),\quad \text{(B5)}$$

$$\{j_k'(x,t),j_l'(x',t')\}=[j_l'(x,t)\nabla_k\delta(x-x') - j_k'(x',t')\nabla_l\delta(x-x')]\delta(t-t').$$

All other Poisson brackets are zero. The reversible terms (B3) of the dynamic equations turn with (B5) to (we omit the explicit indication of space and time dependence in the following)

$$V_s=-k_BT\boldsymbol{\nabla}\cdot\left(s\frac{\delta H}{\delta\boldsymbol{j}'}\right),\qquad \text{(B6)}$$

$$V_\rho=-k_BT\boldsymbol{\nabla}\cdot\left(\rho\frac{\delta H}{\delta\boldsymbol{j}'}\right),\qquad \text{(B7)}$$

$$\boldsymbol{V}_j=-k_BT\left[s\boldsymbol{\nabla}\frac{\delta H}{\delta s}+\rho\boldsymbol{\nabla}\frac{\delta H}{\delta\rho}\right]-k_BT\sum_k\left[j_k'\boldsymbol{\nabla}\frac{\delta H}{\delta j_k'} -\nabla_k\boldsymbol{j}'\frac{\delta H}{\delta j_k'}\right].\qquad \text{(B8)}$$

The matrix Λ_{ij} is determined by the dissipation processes in hydrodynamics. For a liquid at rest ($\boldsymbol{v}=0$) the hydrodynamic equation for the entropy density reads [16]

$$T\frac{\partial s}{\partial t}=-\boldsymbol{\nabla}\cdot\boldsymbol{q},\quad \boldsymbol{q}=-\kappa_T^{(0)}\boldsymbol{\nabla}T\qquad \text{(B9)}$$

in which $\kappa_T^{(0)}$ is the thermal conductivity in the background. Expanding the functional H in Eq. (A6) in powers of the fluctuations $\Delta s(x)$, $\Delta\rho(x)$ and $\Delta\boldsymbol{j}'(x)$, a comparison of the coefficients in quadratic order with thermodynamic relations show that we can write $\nabla^2T=k_BT\,\delta H/\delta s$. Thus the hydrodynamic equation (B9) can be written as

$$\frac{\partial s}{\partial t}=k_B\kappa_T^{(0)}\nabla^2\frac{\delta H}{\delta s}.\qquad \text{(B10)}$$

From the above equation it follows that in the dynamic equation (B1) for the entropy density the only nonvanishing kinetic coefficient is $\Lambda_{ss}=-k_B\kappa_T^{(0)}\nabla^2$. With Eq. (B6) we get

$$\frac{\partial s}{\partial t}=k_B\kappa_T^{(0)}\nabla^2\frac{\delta H}{\delta s}-k_BT\boldsymbol{\nabla}\cdot\left(s\frac{\delta H}{\delta\boldsymbol{j}'}\right)+\Theta_s\qquad \text{(B11)}$$

for the nonlinear entropy density equation. Due to mass conservation no dissipative contributions appear in the dynamic equation for the mass density. With Eq. (B7) we simply get

$$\frac{\partial\rho}{\partial t}=-k_BT\boldsymbol{\nabla}\cdot\left(\rho\frac{\delta H}{\delta\boldsymbol{j}'}\right).\qquad \text{(B12)}$$

The conservation of mass is an exact relation, therefore Eq. (B12) contains no stochastic force. Linearizing the hydrodynamic equation for the momentum density in the velocity, the equation reads [16]

$$\frac{\partial\boldsymbol{j}'}{\partial t}=\left(\zeta^{(0)}+\frac{\overline{\eta^{(0)}}}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot v)+\overline{\eta^{(0)}}\nabla^2\boldsymbol{v}\qquad \text{(B13)}$$

$\zeta^{(0)}$ and $\overline{\eta^{(0)}}$ are the bulk viscosity and the shear viscosity in the noncritical background. Equation (B13) may be separated into an equation for the longitudinal and transverse parts of the momentum density according to $\boldsymbol{j}'=\boldsymbol{j}'_l+\boldsymbol{j}'_t$ with $\boldsymbol{\nabla}\times\boldsymbol{j}'_l=\boldsymbol{0}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{j}'_t=0$:

$$\frac{\partial\boldsymbol{j}'_l}{\partial t}=\left(\zeta^{(0)}+\frac{4}{3}\overline{\eta^{(0)}}\right)\nabla^2\boldsymbol{v}_l,\qquad \frac{\partial\boldsymbol{j}'_t}{\partial t}=\overline{\eta^{(0)}}\nabla^2\boldsymbol{v}_t.\quad \text{(B14)}$$

From the kinetic energy in the static functional (A6) it follows that the longitudinal and transverse velocity in Eq. (B14) can be written as $\boldsymbol{v}_i=k_BT\,\delta H/\delta\boldsymbol{j}'_i$ $(i=l,t)$. With the reversible term (B8) we get for the nonlinear dynamic equation

$$\frac{\partial\boldsymbol{j}'}{\partial t}=k_BT\left(\zeta^{(0)}+\frac{4}{3}\overline{\eta^{(0)}}\right)\nabla^2\frac{\delta H}{\delta\boldsymbol{j}'_l}+k_BT\,\overline{\eta^{(0)}}\nabla^2\frac{\delta H}{\delta\boldsymbol{j}'_t} -k_BT\left[s\boldsymbol{\nabla}\frac{\delta H}{\delta s}+\rho\boldsymbol{\nabla}\frac{\delta H}{\delta\rho}\right]-k_BT\sum_k\left[j_k'\boldsymbol{\nabla}\frac{\delta H}{\delta j_k'} -\nabla_k\boldsymbol{j}'\frac{\delta H}{\delta j_k'}\right]+\boldsymbol{\Theta}_{j'}.\qquad \text{(B15)}$$

Changing from entropy per volume to entropy per mass $\sigma(x)=s(x)/\rho(x)$, analogous to statics, Eq. (B11) turns into

$$\frac{\partial\sigma}{\partial t}=\frac{k_B\kappa_T^{(0)}}{\rho^2}\nabla^2\frac{\delta H}{\delta\sigma}-k_BT(\boldsymbol{\nabla}\sigma)\cdot\frac{\delta H}{\delta\boldsymbol{j}'}+\Theta_\sigma.\quad \text{(B16)}$$

From Eq. (B15) we get the corresponding equation for the momentum density:

$$\frac{\partial\boldsymbol{j}'}{\partial t}=k_BT\left(\zeta^{(0)}+\frac{4}{3}\overline{\eta^{(0)}}\right)\nabla^2\frac{\delta H}{\delta\boldsymbol{j}'_l}+k_BT\,\overline{\eta^{(0)}}\nabla^2\frac{\delta H}{\delta\boldsymbol{j}'_t} -k_BT\left[\rho\boldsymbol{\nabla}\frac{\delta H}{\delta\rho}-(\nabla\sigma)\frac{\delta H}{\delta\sigma}\right]-k_BT\sum_k\left[j_k'\boldsymbol{\nabla}\frac{\delta H}{\delta j_k'} -\nabla_k\boldsymbol{j}'\frac{\delta H}{\delta j_k'}\right]+\boldsymbol{\Theta}_{j'}.\qquad \text{(B17)}$$

The equation for the mass density (B12) remains unchanged. Equations (B12), (B16), and (B17) constitute a set of nonlinear equations that describe the dynamics of fluctuations in liquids. Equations (2.1)–(2.4) are obtained by introducing the fields ϕ_0 and q_0 from Eqs. (A14) and (A15) and by splitting the momentum density equation into a longitudinal and a transverse part.

APPENDIX C: DYNAMIC FUNCTIONAL

In order to calculate the dynamic correlation functions in a perturbation expansion we need a generating dynamic functional. Considering the dynamic equations (B16) and (B17), we write the equations that contain stochastic forces in a short notation

$$\partial_t \vec{\alpha} = \vec{V} + \vec{\Theta} \quad \text{with} \quad \vec{\alpha} = \begin{pmatrix} \sigma \\ \boldsymbol{j}'_l \\ \boldsymbol{j}'_t \end{pmatrix}, \quad \vec{\Theta} = \begin{pmatrix} \Theta_\sigma \\ \boldsymbol{\Theta}'_l \\ \boldsymbol{\Theta}'_t \end{pmatrix}. \tag{C1}$$

The vector $\vec{V}$ contains the rest of Eqs. (B16) and (B17). The fluctuating forces $\vec{\Theta}$ fulfill Einstein relations analogous to (2.5) but with a coefficient matrix

$$\boldsymbol{L}' = \begin{pmatrix} -(k_B \kappa_T^{(0)}/\rho^2)\nabla^2 & 0 & 0 \\ 0 & -k_B T\left(\zeta^{(0)} + \frac{4}{3}\overline{\eta^{(0)}}\right)\nabla^2 & 0 \\ 0 & 0 & -k_B T\,\overline{\eta^{(0)}}\nabla^2 \end{pmatrix}. \tag{C2}$$

Additionally we have the exact continuity equation (B12) in the short form

$$\partial_t \rho = V_\rho , \tag{C3}$$

which may be considered as a constraint for the generating functional. The stochastic forces fluctuate in such a way that Eq. (C3) is always fulfilled. Thus the generating functional can be written as

$$Z_d = \int \mathcal{D}(\vec{\Theta})\mathcal{D}(F)\,\delta(F)\exp\left[-\frac{1}{4}\int dt dx \vec{\Theta}^T \boldsymbol{L}'^{-1}\vec{\Theta}\right], \tag{C4}$$

where $F = \partial_t \rho - V_\rho$. $\mathcal{D}$ refers to a suitable integration measure. Inserting Eq. (C1) and changing the integration variables leads to

$$Z_d = \int \mathcal{D}(\vec{\alpha},\rho)\,\delta(\partial_t\rho - V_\rho)\exp\left[-\frac{1}{4}\int dt \int dx\left([\partial_t\vec{\alpha} - \vec{V}]^T \boldsymbol{L}'^{-1}[\partial_t\vec{\alpha} - \vec{V}] + 2\sum_i \frac{\delta V_i}{\delta \alpha_i} + 2\frac{\delta V_\rho}{\delta \rho}\right)\right]. \tag{C5}$$

The δ function may be expressed by an exponential function

$$\delta(\partial_t\rho - V_\rho) = \int \mathcal{D}(i\tilde{\rho})\exp\left[-\int dx \int dt \tilde{\rho}(\partial_t\rho - V_\rho)\right]. \tag{C6}$$

Introducing auxiliary fields $i\tilde{\vec{\alpha}}$ and performing a Gaussian transformation (C5) turns into

$$Z_d = \int \mathcal{D}(\vec{\alpha},\rho,i\tilde{\vec{\alpha}},i\tilde{\rho})e^{-J} \tag{C7}$$

with

$$J = \int dt \int dx\left(-\tilde{\vec{\alpha}}^T \boldsymbol{L}'\tilde{\vec{\alpha}} + \tilde{\vec{\alpha}}^T(\partial_t\vec{\alpha} - \vec{V}) + \tilde{\rho}(\partial_t\rho - V_\rho) + \frac{1}{2}\sum_i \frac{\delta V_i}{\delta\alpha_i} + \frac{1}{2}\frac{\delta V_\rho}{\delta\rho}\right). \tag{C8}$$

Introducing the order parameter (A14) and the secondary density (A15) in Eq. (C8) the dynamic functional reads

$$J = \int dt \int dx\left(-\tilde{\vec{\beta}}^T \boldsymbol{L}\tilde{\vec{\beta}} + \tilde{\vec{\beta}}^T(\partial_t\vec{\beta} - \vec{V}) + \frac{1}{2}\sum_i \frac{\delta V_i}{\delta\beta_i}\right), \tag{C9}$$

where the densities are $\vec{\beta}^T = (\phi_0, q_0, \boldsymbol{j}_l, \boldsymbol{j}_t)$ and $\boldsymbol{L}$ is the coefficient matrix (2.6). The conjugated densities $\tilde{\vec{\beta}}$ are defined accordingly. An explicit expression for (C9) is obtained by inserting the dynamic equations (2.1)–(2.4). The Fourier transformed Gaussian part can be written as

$$J^{(0)}=\frac{1}{2}\int_{k,\omega}[\vec{\beta}^T(k,\omega),\tilde{\vec{\beta}}^T(k,\omega)]\boldsymbol{\Gamma}^{(0)}(k,\omega)\begin{pmatrix}\vec{\beta}(-k,-\omega)\\ \tilde{\vec{\beta}}(-k,-\omega)\end{pmatrix}. \tag{C10}$$

The integration is defined as $\int_{k,\omega}=\int[d^dk/(2\pi)^d]\int d\omega/2\pi$. The elements of the matrix $\boldsymbol{\Gamma}^{(0)}(k,\omega)$ are the dynamic vertex functions in lowest order perturbation theory. They are explicitly given by

$$\boldsymbol{\Gamma}^{(0)}(k,\omega)=\begin{pmatrix}\mathbf{0} & -i\omega\mathbf{1}+\boldsymbol{L}(k)\\ i\omega\mathbf{1}+\boldsymbol{L}^{\dagger}(k) & -2\boldsymbol{\lambda}(k)\end{pmatrix}, \tag{C11}$$

where $\mathbf{1}$ denotes the unit matrix and the superscript † the adjoint matrix. In the present case the submatrices are

$$\boldsymbol{L}(k)=\begin{pmatrix}\mathring{\Gamma}k^2(\mathring{\tau}+k^2) & \mathring{L}_{\phi}k^2(\mathring{\tau}+k^2) & -ik\mathring{g}_l\mathring{h}_q & 0\\ a_q\mathring{L}_{\phi}k^2 & a_q\mathring{\lambda}k^2 & ika_q\mathring{c} & 0\\ 0 & ika_j\mathring{c} & a_j\mathring{\lambda}_lk^2 & 0\\ 0 & 0 & 0 & a_j\mathring{\lambda}_tk^2\end{pmatrix}, \tag{C12}$$

$$\boldsymbol{\lambda}(k)=\begin{pmatrix}\mathring{\Gamma}k^2 & \mathring{L}_{\phi}k^2 & 0 & 0\\ \mathring{L}_{\phi}k^2 & \mathring{\lambda}k^2 & 0 & 0\\ 0 & 0 & \mathring{\lambda}_lk^2 & 0\\ 0 & 0 & 0 & \mathring{\lambda}_tk^2\end{pmatrix}. \tag{C13}$$

The interaction terms in the Hamiltonian (2.10) and the mode coupling terms in the dynamic equation modify the matrix (C11) and may be calculated in a perturbation expansion. The dynamic two-point vertex function is given by

$$\boldsymbol{\Gamma}(k,\omega)=\boldsymbol{\Gamma}^{(0)}(k,\omega)-\boldsymbol{\Sigma}(k,\omega), \tag{C14}$$

where $\boldsymbol{\Sigma}(k,\omega)$ contains 1-irreducible diagrams with two external legs. The matrix $\boldsymbol{\Gamma}(k,\omega)$ of the vertex functions has the structure

$$\boldsymbol{\Gamma}(k,\omega)=\begin{pmatrix}[0] & [\Gamma_{\alpha,\tilde{\beta}}](k,\omega)\\ [\Gamma_{\tilde{\alpha},\beta}](k,\omega) & [\Gamma_{\tilde{\alpha},\tilde{\beta}}](k,\omega)\end{pmatrix}, \tag{C15}$$

with the submatrix

$$[\Gamma_{\alpha\tilde{\beta}}]=\begin{pmatrix}\mathring{\Gamma}_{\phi\tilde{\phi}} & \mathring{\Gamma}_{\phi\tilde{q}} & \mathring{\Gamma}_{\phi\tilde{l}} & 0\\ \mathring{\Gamma}_{q\tilde{\phi}} & \mathring{\Gamma}_{q\tilde{q}} & \mathring{\Gamma}_{q\tilde{l}} & 0\\ \mathring{\Gamma}_{l\tilde{\phi}} & \mathring{\Gamma}_{l\tilde{q}} & \mathring{\Gamma}_{q\tilde{l}} & 0\\ 0 & 0 & 0 & \mathring{\Gamma}_{t\tilde{t}}\end{pmatrix}. \tag{C16}$$

The submatrices $[\Gamma_{\tilde{\alpha},\beta}]$ and $[\Gamma_{\tilde{\alpha},\tilde{\beta}}]$ are defined accordingly. Then the propagators of the model are determined by inverting Eq. (C11). In the limit $\mathring{c}\to\infty$ the propagators of order $(\mathring{c})^0$ are identical to the known model H propagators [1]. One gets the response propagators

$$\langle\phi_0(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle_0=\frac{1}{-i\omega+\mathring{\Gamma}k^2(\mathring{\tau}+k^2)}, \tag{C17}$$

$$\langle\boldsymbol{j}_t(k,\omega)\otimes\tilde{\boldsymbol{j}}_t(-k,-\omega)\rangle_0=\frac{1}{-i\omega+a_j\mathring{\lambda}_tk^2}\mathbf{1}, \tag{C18}$$

and the correlation propagators

$$\langle\phi_0(k,\omega)\phi_0(-k,-\omega)\rangle_0=\frac{2\mathring{\Gamma}k^2}{|-i\omega+\mathring{\Gamma}k^2(\mathring{\tau}+k^2)|^2}, \tag{C19}$$

$$\langle\boldsymbol{j}_t(k,\omega)\otimes\boldsymbol{j}_t(-k,-\omega)\rangle_0=\frac{2\mathring{\lambda}_tk^2}{|-i\omega+a_j\mathring{\lambda}_tk^2|^2}\mathbf{1}. \tag{C20}$$

In the extended model additional propagators of order $(\mathring{c})^{-1}$ arise, which contribute in the limit $\mathring{c}\to\infty$ with vertices of order $\mathring{c}$ to the vertex functions, they read

$$\langle\phi_0(k,\omega)\tilde{\boldsymbol{j}}_l(-k,-\omega)\rangle_0=-\frac{\mathring{L}_{\phi}\boldsymbol{k}}{i\mathring{c}[-i\omega+\mathring{\Gamma}k^2(\mathring{\tau}+k^2)]}, \tag{C21}$$

$$\langle\boldsymbol{j}_l(k,\omega)\tilde{\phi}_0(-k,-\omega)\rangle_0=-\frac{\mathring{L}_{\phi}(\mathring{\tau}+k^2)\boldsymbol{k}}{ia_j\mathring{c}[-i\omega+\mathring{\Gamma}k^2(\mathring{\tau}+k^2)]}, \tag{C22}$$

$$\langle q_0(k,\omega)\tilde{\phi}(-k,-\omega)\rangle_0=-\frac{\mathring{g}_l\mathring{h}_q}{ia_q\mathring{c}[-i\omega+\mathring{\Gamma}k^2(\mathring{\tau}+k^2)]}, \tag{C23}$$

$$\langle q_0(k,\omega)\tilde{\boldsymbol{j}}_l(-k,-\omega)\rangle_0=\frac{\boldsymbol{k}}{ia_q\mathring{c}k^2}, \tag{C24}$$

$$\langle\boldsymbol{j}_l(k,\omega)\tilde{q}_0(-k,-\omega)\rangle_0=\frac{\boldsymbol{k}}{ia_j\mathring{c}k^2}, \tag{C25}$$

$$\langle\phi_0(k,\omega)q_0(-k,-\omega)\rangle_0=\frac{2\mathring{\Gamma}k^2\mathring{g}_l\mathring{h}_q}{a_q\mathring{c}|-i\omega+\mathring{\Gamma}k^2(\mathring{\tau}+k^2)|^2}, \tag{C26}$$

$$\langle\phi_0(k,\omega)\boldsymbol{j}_l(-k,-\omega)\rangle_0=-\frac{2\mathring{L}_{\phi}\boldsymbol{k}\omega}{a_j\mathring{c}|-i\omega+\mathring{\Gamma}k^2(\mathring{\tau}+k^2)|^2}. \tag{C27}$$

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